

Asymptotic behaviour and the Nahm transform of doubly periodic instantons with square integrable curvature

Takuro Mochizuki

Abstract

We study the asymptotic behaviour of doubly periodic instantons with square-integrable curvature. Then, we establish the equivalence given by the Nahm transform between the doubly periodic instantons with square integrable curvature and the wild harmonic bundles on the dual torus.

1 Introduction

Let $T := \mathbb{C}/L$, where L is a lattice of \mathbb{C} . The product $T \times \mathbb{C}$ is equipped with the standard metric $dz d\bar{z} + dw d\bar{w}$, where (z, w) is the standard local coordinate of $T \times \mathbb{C}$. In this paper, we shall study an L^2 -instanton (E, ∇, h) on $T \times \mathbb{C}$, i.e., the curvature $F(\nabla)$ satisfies the equation $\Lambda F(h) = 0$, and it is L^2 .

There is another natural decay condition around ∞ . That is the quadratic curvature decay, i.e., $|F(\nabla)|_h = O(|w|^{-2})$. M. Jardim [26] studied the Nahm transform of some kind of tame harmonic bundles on the dual torus T^\vee to produce instantons on $T \times \mathbb{C}$ satisfying the quadratic curvature decay. O. Biquard and Jardim [8] studied the asymptotic behaviour of such instantons with rank 2. Based on the results, the inverse transform was constructed in [27], i.e., the Nahm transform of such instantons on $T \times \mathbb{C}$ to produce some type of tame harmonic bundles on T^\vee . See also [28] and [29].

It is our purpose in this paper to generalize their results. Namely, we will study the asymptotic behaviour of L^2 -instantons, and establish the equivalence between the L^2 -instantons on $T \times \mathbb{C}$ and wild harmonic bundles on T^\vee .

1.1 Asymptotic behaviour of L^2 -instanton

1.1.1 The dimensional reduction due to Hitchin

Briefly speaking, our goal is to show that an L^2 -instanton behaves like a wild harmonic bundle around ∞ . For the explanation, let us recall the dimensional reduction due to Hitchin. Let U be an open subset of \mathbb{C} . Let $(V, \bar{\partial}_V)$ be a holomorphic vector bundle on U with a Higgs field θ . Let h be a hermitian metric of V . We have the Chern connection $\nabla_{V,h} = \bar{\partial}_V + \partial_{V,h}$. We have the adjoint θ^\dagger of θ with respect to h . The tuple $(V, \bar{\partial}_V, h, \theta)$ is called a harmonic bundle, if the Hitchin equation $F(\nabla_{V,h}) + [\theta, \theta^\dagger] = 0$ is satisfied.

Let $p : T \times U \rightarrow U$ be the projection. We have the expression $\theta = f dw$ and $\theta^\dagger = f d\bar{w}$, where f is a holomorphic endomorphism of V , and f^\dagger is the adjoint of f . We set $(E, h_E) := p^*(V, h)$. Let ∇_E be the unitary connection given by $\nabla_E = p^*(\nabla_{V,h}) + f d\bar{z} - f^\dagger dz$. Then, (E, h_E, ∇_E) is an instanton, if and only if $(V, \bar{\partial}_V, h, \theta)$ is a harmonic bundle. Indeed, we have the equivalence between harmonic bundles on U , and T -equivariant instantons on $T \times U$, which is due to N. Hitchin.

1.1.2 Example and Remarks

We set $U := \{w \in \mathbb{C} \mid |w| > R\}$. We shall make R larger without mention. Let \mathbf{a} be any holomorphic function on U . We have the harmonic bundle $\mathcal{L}(\mathbf{a})$ obtained as the tuple of the trivial line bundle $\mathcal{O}_U e$, the trivial metric $h(e, e) = 1$ and the Higgs field $d\mathbf{a}$. By using the dimensional reduction, we have the associated instanton. Its curvature is $\partial_w^2 \mathbf{a} dw d\bar{z} + \bar{\partial}_w^2 \mathbf{a} d\bar{z} d\bar{w}$, which is L^2 if and only if it has quadratic decay.

We can obtain more examples by considering a ramification along ∞ . We set $U_\eta := \{\eta \in \mathbb{C} \mid |\eta| > R^{1/2}\}$. We consider a harmonic bundle $\mathcal{L}(\mathbf{a})$, where \mathbf{a} is a holomorphic function on U_η . Let $\varphi : U_\eta \rightarrow U$ be given by

$\varphi(\eta) = \eta^2$. We obtain a harmonic bundle $\varphi_*\mathcal{L}(\mathbf{a})$ on U obtained as the push-forward. It is easy to check that the associated instanton is L^2 if and only if $\eta^{-2}\mathbf{a}(\eta)$ is holomorphic at ∞ . In that case, the curvature F satisfies the decay condition $O(|w|^{-3/2})$. If $\mathbf{a} = \alpha\eta$ for $\alpha \neq 0$, we have $0 < C_1 < |F| |w|^{3/2} < C_2$ for some constants C_i .

More generally, for any positive integer p , we set $U^{(p)} := \{w_p \in \mathbb{C} \mid |w_p| > R^{1/p}\}$. For a covering $\varphi_p : U^{(p)} \rightarrow U$ given by $\varphi_p(w_p) = w_p^p$ and for a holomorphic function \mathbf{a} on $U^{(p)}$, we obtain a harmonic bundle $\varphi_{p*}\mathcal{L}(\mathbf{a})$ on U . The associated instanton is L^2 if and only if $\varphi^*(w)^{-1}\mathbf{a}$ is holomorphic at ∞ . If \mathbf{a} is a polynomial of w_p , then it is described as a condition on the slope $\deg_{w_p}(\mathbf{a})/p \leq 1$. In that case, the curvature F satisfies $O(|w|^{-1-1/p})$. It is easy to construct an example satisfying $0 < C_1 < |F| |w|^{1+1/p} < C_2$ for some $C_i > 0$.

Let $(E, \bar{\partial}_E, \theta, h)$ be a wild harmonic bundle on U . As the above examples suggests, the L^2 -condition and the quadratic decay condition can be described in terms of the eigenvalues of the Higgs field. If we take an appropriate covering $\varphi_p : U^{(p)} \rightarrow U$, we have a holomorphic decomposition

$$\varphi_p^*(E, \theta) = \bigoplus_{\mathbf{a} \in w_p \mathbb{C}[w_p]} (E_{\mathbf{a}}, \theta_{\mathbf{a}}) \quad (1)$$

such that $\theta_{\mathbf{a}} - d\mathbf{a}$ are tame. The harmonic bundle is called unramified, if $(E, \bar{\partial}_E, \theta, h)$ itself has a decomposition such as (1). We set $\text{Irr}(\theta) := \{\mathbf{a} \mid E_{\mathbf{a}} \neq 0\}$. Then, by using the results in the asymptotic behaviour of wild harmonic bundles [38], it is not difficult to show that the instanton associated to $(E, \bar{\partial}_E, \theta, h)$ is L^2 , if and only if $\deg_{w_p}(\mathbf{a})/p \leq 1$ for any $\mathbf{a} \in \text{Irr}(\theta)$. It satisfies the quadratic decay condition, if and only if the harmonic bundle is unramified, in addition.

The condition can also be described in terms of the spectral variety of θ . We have the expression $\theta = f dw$. Let $\mathcal{S}p(f) \subset \mathbb{C}_{\zeta} \times U$ denote the support of the cokernel of $\mathcal{O}_{\mathbb{C}_{\zeta} \times U} \xrightarrow{\zeta - f} \mathcal{O}_{\mathbb{C}_{\zeta} \times U}$. It induces a subvariety $\Phi(\mathcal{S}p(f))$, where $\Phi : \mathbb{C}_{\zeta} \times U \rightarrow T^{\vee} \times U$ be the projection. Then, the instanton associated to $(E, \bar{\partial}_E, \theta, h)$ is L^2 , if and only if the closure of $\Phi(\mathcal{S}p(f))$ in $T \times \bar{U}$ is a complex subvariety, where $\bar{U} = U \cup \{\infty\}$.

1.1.3 Brief description of the asymptotic behaviour of L^2 -instantons

Let (E, ∇, h) be an L^2 -instanton on $T \times U$. Let $(E, \bar{\partial}_E)$ denote the underlying holomorphic vector bundle on $T \times U$. By using a theorem of Uhlenbeck, we obtain $F(\nabla) = o(|w|^{-1})$. It implies that the restrictions $(E, \bar{\partial}_E)|_{T \times \{w\}}$ are semistable of degree 0. Hence, the relative Fourier-Mukai transform of $(E, \bar{\partial}_E)$ gives an $\mathcal{O}_{T^{\vee} \times U}$ -module whose support $\mathcal{S}p(E)$ is relatively 0-dimensional over U . The first important issue in the study is to establish that $\mathcal{S}p(E)$ is extended to a subvariety $\overline{\mathcal{S}p(E)}$ in $T \times \bar{U}$ (Theorem 3.13). We use an effective comparison of the spectrum of semistable bundles of degree 0 and the eigenvalues of the monodromy transformations of unitary connections with the small curvature (Corollary 2.13), and a result for extensions of a holomorphic map on a punctured disc to a projective space (Theorem 3.21).

Let $\pi : T \times U \rightarrow U$ denote the projection. We fix a lift of $\overline{\mathcal{S}p(E)}$ to $\widetilde{\mathcal{S}p(E)} \subset \mathbb{C} \times \bar{U}$. Then, we obtain a holomorphic vector bundle V on U with an endomorphism g , with an isomorphism $\pi^*V \simeq E$ such that (i) $\pi^*\bar{\partial}_V + g d\bar{z} = \bar{\partial}_E$, (ii) $\mathcal{S}p(g) = \widetilde{\mathcal{S}p(E)}$. By the identification $E = \pi^*V$, we obtain a T -action on E . By using the Fourier expansion, we decompose h into the T -invariant part and the complement. The T -invariant part naturally gives a hermitian metric h_V of V . We shall prove that the complement and its derivatives have exponential decay (Theorem 3.14), and hence $(V, \bar{\partial}_V, h_V, g dw)$ satisfies the Hitchin equation up to an exponentially small term (Proposition 3.16).

Such a tuple $(V, \bar{\partial}_V, h_V, g dw)$ can be studied as in the case of wild harmonic bundles [38] with minor modifications. (See §3.6.) Thus, we will arrive at a satisfactory stage of understanding of the asymptotic behaviour of L^2 -instantons. For example, we obtain that the curvature has the decay $O(|w|^{-1-\rho})$ for some $\rho > 0$ with respect to h and the Euclidean metric (Theorem 3.17). It implies that $(E, \bar{\partial}_E, h)$ is acceptable, i.e., $F(\nabla)$ is bounded with respect to h and the Poincaré like metric on $T \times U$. Hence, the holomorphic bundle $(E, \bar{\partial}_E)$ is naturally extended to a filtered bundle \mathcal{P}_*E on $(T \times \bar{U}, T \times \{\infty\})$.

We also observe that, we need only a weaker assumption on the curvature decay, if we assume the prolongation of the spectral curve. (See §3.4.)

In [8], Jardim and Biquard showed that an instanton of rank 2 with quadratic decay is a exponentially small perturbation of $(V, \bar{\partial}_V, g dw, h_V)$, which satisfies the Hitchin equation up to an exponentially small term.

Our result could be regarded as a generalization of theirs. But, the methods are rather different. To obtain a decomposition into the T -invariant part and the complement, they started with the construction of a global frame satisfying some nice property, which is an analogue of the Coulomb gauge of K. Uhlenbeck. Their method seems to require a stronger decay condition than L^2 , for example the quadratic decay condition as they imposed. We use a more natural decomposition induced by a standard method of the Fourier-Mukai transform in complex geometry, which allows us to consider L^2 -instantons, once we deal with the issue of the prolongation of the spectral curve. (See §3.3.)

As mentioned above, we shall establish that an L^2 -instanton is an exponentially small perturbation of $(V, \bar{\partial}_V, h_V, \theta_V)$ which satisfies the Hitchin equation up to exponentially small term. Interestingly to the author, we can obtain a more refined result. Namely, we can naturally construct a Hermitian-Einstein metric h'_V of $(V, \bar{\partial}_V, g dw)$ defined on a neighbourhood of ∞ , from the L^2 -instanton. It is an analogue of the reductions from wild harmonic bundles to tame harmonic bundles studied in [38]. We consider a kind of meromorphic prolongation of the holomorphic vector bundle on the twistor space associated to $T \times \mathbb{C}$, then we encounter a kind of infinite dimensional Stokes phenomena. By taking the graduation with respect to the Stokes structure, we obtain a wild harmonic bundle. Relatedly, in this paper, we consider only the product holomorphic structure of $T \times \mathbb{C}$. From the viewpoint of twistor theory, the holomorphic vector bundle with respect to the other holomorphic structures should also be studied. The prolongation of the twistor family of the holomorphic structure is related with the issue in the previous paragraph. The author hopes to return to this deeper aspect of the study elsewhere.

1.2 Nahm transform for wild harmonic bundles and L^2 -instantons

As an application of the study of the asymptotic behaviour, we shall establish the equivalence between L^2 -instantons on $T \times \mathbb{C}$ and wild harmonic bundles on T^\vee given by the Nahm transforms, which is a differential geometric variant of the Fourier-Mukai transform. (See [4] and [29] for the long history of a various versions of the Nahm transforms.)

Once we understand the asymptotic behaviour, the standard method allows us to construct the Nahm transform of L^2 -instantons of $T \times \mathbb{C}$ to wild harmonic bundles on T^\vee . (See §4.4.) We can also construct the Nahm transform of wild harmonic bundles on T^\vee to L^2 -instantons on $T \times \mathbb{C}$, by using the result on wild harmonic bundles on curves ([38], [42] and [53]), although we need some estimates to establish the L^2 -property (see §6.1).

To study more detailed properties of the Nahm transform, we introduce the algebraic Nahm transforms for admissible filtered Higgs fields on (T^\vee, D) and admissible filtered bundles on $(T \times \mathbb{C}, T \times \{\infty\})$, based on the Higgs interpretation of the Nahm transform. (See §5.) It could be regarded as a filtered version of the Fourier transform for Higgs bundles studied in [9], although we restrict ourselves to the case that the base space is an elliptic curve.

We begin with the local versions in §5.1, i.e., the local algebraic Nahm transforms $\mathcal{N}^{0,\infty}$ and $\mathcal{N}^{\infty,0}$ for admissible filtered Higgs bundles, which are analogue of the local Fourier transform for D -modules on \mathbb{P}^1 in [14] and [21]. They give an equivalence between the category of germs of admissible filtered Higgs bundles and the category of germs of admissible filtered bundles. Although we also give a refinement for good filtered Higgs bundles and good filtered bundles in §5.4, it seems easier to begin with the transforms for admissible ones.

We define in §5.2 the algebraic Nahm transform from admissible filtered Higgs bundles on (T^\vee, D) to a meromorphic bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$, by using the holomorphic counterpart of the L^2 -cohomology of wild harmonic bundles in [38]. (It goes back to the theory of the L^2 -cohomology groups of singular variation of Hodge structures [53].) We use the local algebraic Nahm transform to enhance it with filtrations. Similarly, we define in §5.3 the algebraic Nahm transform for admissible filtered bundles by using the holomorphic counterpart of the L^2 -cohomology in §4.2, and enhance it with filtrations by using the local algebraic Nahm transform.

We establish the compatibility of the Nahm transform and the algebraic Nahm transform (Theorem 6.11 and Theorem 6.12). Namely, the algebraic Nahm transform describes the correspondences between filtered Higgs bundles on T^\vee and filtered bundles on $T \times \mathbb{P}^1$ induced by the Nahm transform. It is rather obvious to obtain the compatibility in the smooth part, which implies the comparison of the spectral curves. We need some estimates for the comparison of the parabolic structure, together with the comparison of the characteristic numbers.

We use the compatibility to prove the inversion. For the algebraic Nahm transforms, it is rather obvious that the constructions are mutually inverse. (See Proposition 5.21.) Then, we can deduce the inversion formula for the Nahm transforms (Corollary 6.13) by using the uniqueness of the Hermitian-Einstein metrics (resp. the harmonic metrics) adapted to the filtered bundles (resp. filtered Higgs bundles).

As other applications of the compatibility, we can easily compute the characteristic classes of the bundles obtained as the algebraic Nahm transform, which allows us to obtain the rank and the second Chern class of the bundle obtained as the Nahm transform. The local algebraic Nahm transform leads us a rather complete understanding of the transformation of singularity data by the Nahm transform.

Remark 1.1 *If we consider a counterpart of the algebraic Nahm transform for the other non-product holomorphic structure of $T \times \mathbb{C}$ underlying the hyperKähler structure, it is essentially a filtered version of the generalized Fourier-Mukai transform in [34] and [41]. Interestingly to the author, we have an analogue of the stationary phase formula even in this case. The details will be given elsewhere.* ■

Remark 1.2 *In this paper, we consider transforms between filtered bundles on $T \times \mathbb{P}^1$ and filtered Higgs bundles on T^\vee . We may introduce similar transforms for filtered Higgs bundles on \mathbb{P}^1 with an additional work on the local Nahm transform $\mathcal{N}^{\infty, \infty}$, which is an analogue of the local Fourier transform $\mathcal{F}^{\infty, \infty}$. It should be the Higgs counterpart of the Nahm transforms between wild harmonic bundles on \mathbb{P}^1 , which is given by the procedure for wild pure twistor D -modules established in [38].*

Relatedly, Sz. Szabó [50] studied the Nahm transform for some interesting type of harmonic bundles on \mathbb{P}^1 . He also studied the transform of the underlying parabolic Higgs bundles, which looks closely related with the regular version of ours in §5.2. K. Aker and he [1] introduced a transformation of more general parabolic Higgs bundles on \mathbb{P}^1 , which they call the algebraic Nahm transform. Their method to define the transform is different from ours, and the precise relation is not clear at this moment. ■

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2 Semistable bundles of degree 0 on elliptic curves

2.1 Preliminaries

2.1.1 Elliptic curve and Fourier-Mukai transform

For a variable z , let \mathbb{C}_z denote a complex line with the standard coordinate z . For a \mathbb{C} -vector space V and a C^∞ -manifold X , let \underline{V}_X denote the product bundle $V \times X$ over X . If X is a complex manifold, the natural holomorphic structure of \underline{V}_X is denoted just by $\bar{\partial}$.

We have a real bilinear map $\mathbb{C}_z \times \mathbb{C}_{\bar{z}} \longrightarrow \mathbb{R}$ given by $(z, \bar{z}) \longmapsto \text{Im}(z\bar{z})$. Let $\tau = \tau_1 + \sqrt{-1}\tau_2$ ($\tau_i \in \mathbb{R}$) be a complex number such that $\tau_2 \neq 0$. Let $L := \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}_z$. In this paper, the dual lattice L^\vee means the following:

$$L^\vee := \left\{ \zeta \in \mathbb{C}_{\bar{z}} \mid \text{Im}(\chi\bar{\zeta}) \in \pi\mathbb{Z} \ (\forall \chi \in L) \right\} = \left\{ \frac{\pi}{\tau_2}(n + m\tau) \mid n, m \in \mathbb{Z} \right\}$$

We obtain the elliptic curves $T := \mathbb{C}_z/L$ and $T^\vee := \mathbb{C}_{\bar{z}}/L^\vee$.

For any $\nu \in L^\vee$, we have $\rho_\nu \in C^\infty(T)$ given by $\rho_\nu(z) := \exp(2\sqrt{-1}\text{Im}(\nu\bar{z})) = \exp(\nu\bar{z} - \bar{\nu}z)$. We have $\bar{\partial}_z \rho_\nu = \rho_\nu \nu d\bar{z}$ and $\partial_z \rho_\nu = -\rho_\nu \bar{\nu} dz$.

We can naturally regard T^\vee as the moduli space $\text{Pic}_0(T)$ of holomorphic line bundles of degree 0 on T . Indeed, ζ gives a holomorphic bundle $\mathcal{L}_\zeta = (\underline{\mathbb{C}}_T, \bar{\partial} + \zeta d\bar{z})$. It induces an isomorphism $T^\vee \simeq \text{Pic}_0(T)$. We have the isomorphism $\Phi : \mathcal{L}_\zeta \simeq \mathcal{L}_{\zeta+\nu}$ given by $\Phi(f) = \rho_{-\nu} \cdot f$.

We have the unitary flat connection associated to \mathcal{L}_ζ with the trivial metric, which is $d - \bar{\zeta} dz + \zeta d\bar{z}$. The monodromy along the segment from 0 to $\chi \in L$ is $\exp\left(\int_0^1 (-\bar{\zeta}\chi + \zeta\bar{\chi}) dt\right) = \exp(2\sqrt{-1} \text{Im}(\zeta\bar{\chi}))$.

We recall a differential geometric construction of the Poincaré bundle on $T \times T^\vee$, following [13]. On $T \times \mathbb{C}_\zeta$, we have the holomorphic line bundle $\widetilde{\mathcal{Poin}} = (\underline{\mathbb{C}}_{T \times \mathbb{C}_\zeta}, \bar{\partial} + \zeta d\bar{z})$. The L^\vee -action on $T \times \mathbb{C}_\zeta$ is naturally lifted to the action on $\widetilde{\mathcal{Poin}}$ given by $\nu(z, \zeta, v) = (z, \zeta + \nu, \rho_{-\nu}(z) v)$. Thus, a holomorphic line bundle is induced on $T \times T^\vee$, which is the Poincaré bundle denoted by \mathcal{Poin} . The dual bundle \mathcal{Poin}^\vee is induced by $\widetilde{\mathcal{Poin}}^\vee = (\underline{\mathbb{C}}_{T \times \mathbb{C}_\zeta}, \bar{\partial} - \zeta d\bar{z})$ with the action $\nu(z, \zeta, v) = (z, \zeta + \nu, \rho_\nu(z) v)$.

Let S be any complex analytic space. For $I \subset \{1, 2, 3\}$, let p_I denote the projection of $T \times T^\vee \times S$ onto the product of the i -th components ($i \in I$). For an object $\mathcal{M} \in D^b(\mathcal{O}_{T \times S})$, we set

$$\text{RFM}_\pm(\mathcal{M}) := p_{23*}(p_{13}^*(\mathcal{M}) \otimes p_{12}^*\mathcal{Poin}^{\pm 1})[1] \in D^b(\mathcal{O}_{T^\vee \times S}).$$

For an object $\mathcal{N} \in D^b(\mathcal{O}_{T^\vee \times S})$, we set

$$\widehat{\text{RFM}}_\pm(\mathcal{N}) := p_{13*}(p_{23}^*(\mathcal{N}) \otimes p_{12}^*\mathcal{Poin}^{\pm 1}) \in D^b(\mathcal{O}_{T \times S}).$$

Recall that we have a natural isomorphism $\widehat{\text{RFM}}_+ \circ \text{RFM}_-(\mathcal{M}) \simeq \mathcal{M}$.

2.1.2 Semistable bundle of degree 0

For a holomorphic vector bundle $(E, \bar{\partial}_E)$ on T , we have the degree $\deg(E) := \int_T c_1(E)$ and the slope $\mu(E) := \deg(E)/\text{rank}(E)$. A holomorphic vector bundle E on T is called semistable, if $\mu(F) \leq \mu(E)$ holds for any non-trivial holomorphic subbundle $F \subset E$. In the following, we shall not distinguish a holomorphic vector bundle and the associated sheaf of holomorphic sections.

Let E be a semistable bundle of degree 0 on T . It is well known that the support $\mathcal{S}p(E)$ of $\text{RFM}_-(E)$ are finite points. Indeed, E is obtained as an extension of the line bundles \mathcal{L}_ζ ($\zeta \in \mathcal{S}p(E)$). We call $\mathcal{S}p(E)$ the spectrum of E . We have the decomposition $E = \bigoplus_{\alpha \in \mathcal{S}p(E)} \widetilde{E}_\alpha$, where the support of $\text{RFM}_-(E_\alpha)$ is $\{\alpha\}$. It is called the spectral decomposition of E . We call a subset $\widetilde{\mathcal{S}p}(E) \subset \mathbb{C}$ is a lift of $\mathcal{S}p(E)$, if the projection $\Phi : \mathbb{C} \rightarrow T^\vee$ induces the bijection $\widetilde{\mathcal{S}p}(E) \simeq \mathcal{S}p(E)$. If we fix a lift, an $\mathcal{O}_\mathbb{C}$ -module $\mathcal{M}(E)$ is determined (up to canonical isomorphisms) by the conditions (i) the support of $\mathcal{M}(E)$ is $\widetilde{\mathcal{S}p}(E)$, (ii) $\Phi_*\mathcal{M}(E) \simeq \text{RFM}_-(E)$. Such $\mathcal{M}(E)$ is called a lift of $\text{RFM}_-(E)$. The multiplication of ζ on $\mathcal{M}(E)$ induces an endomorphisms of $\text{RFM}_-(E)$ and E . The endomorphism of E is denoted by f_ζ .

Let S be any complex analytic space. Let E be a holomorphic vector bundle on $T \times S$. It is called semistable of degree 0 relatively to S , if $E|_{T \times \{s\}}$ are semistable of degree 0 for any $s \in S$. The support of $\text{RFM}_-(E)$ is relatively 0-dimensional over S . It is denoted by $\mathcal{S}p(E)$, and called the spectrum of E . If we have a hypersurface $\widetilde{\mathcal{S}p}(E) \subset \mathbb{C}_\zeta \times S$ such that the projection $\Phi : \mathbb{C}_\zeta \times S \rightarrow T^\vee \times S$ induces $\widetilde{\mathcal{S}p}(E) \simeq \mathcal{S}p(E)$, we call $\widetilde{\mathcal{S}p}(E)$ a lift of $\mathcal{S}p(E)$. If we have a lift of $\mathcal{S}p(E)$, we obtain a lift $\mathcal{M}(E)$ of $\text{RFM}_-(E)$ as in the case that S is a point. We also obtain an endomorphism f_ζ of E induced by the multiplication of ζ on $\mathcal{M}(E)$.

2.1.3 Equivalence of categories

For a vector space V , let \underline{V} denote the product bundle $T \times V$ over T , and let $\bar{\partial}_0$ denote the natural holomorphic structure of \underline{V} . For any $f \in \text{End}(V)$, we have the associated holomorphic vector bundle $\mathfrak{G}(V, f) := (\underline{V}, \bar{\partial}_0 + f d\bar{z})$. We have a natural isomorphism $\mathfrak{G}(V, f) \simeq \mathfrak{G}(V, f + \nu \text{id}_V)$ for each $\nu \in L^\vee$, induced by the multiplication of $\rho_{-\nu}$. Let $\mathcal{S}p(f)$ denote the set of the eigenvalues of f .

Lemma 2.1 $\mathfrak{G}(V, f)$ is semistable of degree 0, and we have $\mathcal{S}p(\mathfrak{G}(V, f)) = \Phi(\mathcal{S}p(f))$ in T^\vee , where $\Phi : \mathbb{C} \rightarrow T^\vee$ denotes the projection.

Proof We have only to consider the case f has a unique eigenvalue α . In that case, $\mathfrak{G}(V, f)$ is an extension of the line bundle \mathcal{L}_α . Then, the claim is clear. \blacksquare

Let VS^* denote the category of finite dimensional \mathbb{C} -vector spaces with an endomorphism, i.e., an object in VS^* is a finite dimensional vector space V with an endomorphism f , and a morphism $(V, f) \rightarrow (W, g)$ in VS^* is a linear map $\varphi : V \rightarrow W$ such that $g \circ \varphi - \varphi \circ f = 0$. For a given subset $\tilde{\mathfrak{s}} \subset \mathbb{C}$, let $VS^*(\tilde{\mathfrak{s}}) \subset VS^*$ denote the full subcategory of (V, f) such that $\mathcal{S}p(f) \subset \tilde{\mathfrak{s}}$.

Let $VB_0^{ss}(T)$ denote the category of semistable bundles of degree 0 on T , i.e., an object in $VB_0^{ss}(T)$ is a semistable vector bundle of degree 0 on T , and a morphism $V_1 \rightarrow V_2$ in $VB_0^{ss}(T)$ is a morphism of coherent sheaves. For a given subset $\mathfrak{s} \subset T^\vee$, let $VB_0^{ss}(T, \mathfrak{s}) \subset VB_0^{ss}(T)$ denote the full subcategory of semistable bundles of degree 0 whose spectrum are contained in \mathfrak{s} .

We have the functor $\mathfrak{G} : VS^* \rightarrow VB_0^{ss}(T)$ given by the above construction. If $\tilde{\mathfrak{s}}$ is mapped to \mathfrak{s} by the projection $\Phi : \mathbb{C}_\zeta \rightarrow T^\vee$, it induces a functor $\mathfrak{G} : VS^*(\tilde{\mathfrak{s}}) \rightarrow VB_0^{ss}(T, \mathfrak{s})$.

Proposition 2.2 *If $\Phi : \mathbb{C} \rightarrow T^\vee$ induces a bijection $\tilde{\mathfrak{s}} \simeq \mathfrak{s}$, then \mathfrak{G} gives an equivalence of the categories $VS^*(\tilde{\mathfrak{s}}) \simeq VB_0^{ss}(T, \mathfrak{s})$.*

Proof Let us show that it is fully faithful. We set $E_f := \mathfrak{G}(V, f)$. We will not distinguish E_f and the associated sheaf of holomorphic sections. Suppose that f has a unique eigenvalue α such that $\alpha \not\equiv 0$ modulo L^\vee . Because E_f is obtained as an extension of the holomorphic line bundle \mathcal{L}_α , we have $H^0(T, E_f) = H^1(T, E_f) = 0$. In particular, we obtain the following.

Lemma 2.3 *Assume that $f_i \in \text{End}(V)$ has a unique eigenvalue α_i for $i = 1, 2$. If $\alpha_1 \not\equiv \alpha_2$ modulo L^\vee , any morphism $E_{f_1} \rightarrow E_{f_2}$ is 0.* \blacksquare

Suppose that f is nilpotent. We have the natural inclusion $V \rightarrow C^\infty(T, E_f)$ as constant functions. We have a linear map $V \rightarrow C^\infty(T, E_f \otimes \Omega^{0,1})$ given by $s \mapsto s d\bar{z}$. They induce a chain map ι from $\mathcal{C}_1 = (V \xrightarrow{f} V)$ to the Dolbeault complex $C^\infty(T, E_f \otimes \Omega_T^{0,\bullet})$ of E_f .

Lemma 2.4 *ι is a quasi-isomorphism.*

Proof Let W be the monodromy weight filtration of f . It induces filtrations of \mathcal{C}_1 and $C^\infty(T, E_f \otimes \Omega_T^{0,*})$, and ι gives a morphism of filtered chain complex. It induces a quasi-isomorphism of the associated graded complexes. Hence, ι is a quasi isomorphism. \blacksquare

We obtain the following lemma as an immediate consequence.

Lemma 2.5 *Assume that $f_i \in \text{End}(V)$ are nilpotent ($i = 1, 2$). Then, holomorphic morphisms $E_{f_1} \rightarrow E_{f_2}$ naturally correspond to holomorphic morphisms $\phi : E_0 \rightarrow E_0$ such that $f_2 \circ \phi - \phi \circ f_1 = 0$.*

In particular, if f is nilpotent, holomorphic sections of $\text{End}(E_f)$ bijectively corresponds to holomorphic sections g of $\text{End}(E_0)$ such that $[f, g] = 0$. \blacksquare

The fully faithfulness of the functor \mathfrak{G} follows from Lemma 2.3 and Lemma 2.5. Let us show the essential surjectivity of \mathfrak{G} . Let $E \in VB_0^{ss}(T, \mathfrak{s})$. We have the $\mathcal{O}_{\mathbb{C}_\zeta}$ -module $\mathcal{M}(E)$ and the endomorphism f_ζ of E as in §2.1.2. We have a natural isomorphism $\widehat{\text{RFM}}_+ \circ \text{RFM}_-(E) \simeq E$. The functor $\widehat{\text{RFM}}_+$ is induced by the holomorphic line bundle on $T \times T^\vee$, obtained as the descent of $\widehat{\mathcal{P}oin} = (\mathbb{C}, \bar{\partial}_0 + \zeta d\bar{z})$. Let p and q denote the projections $T \times \mathbb{C}_\zeta \rightarrow T$ and $T \times \mathbb{C}_\zeta \rightarrow \mathbb{C}_\zeta$. We have $E \simeq p_*(q^*(\mathcal{M}) \otimes \widehat{\mathcal{P}oin})$, and the latter is naturally isomorphic to $(H^0(\mathbb{C}_\zeta, \mathcal{M}(E)), \bar{\partial}_0 + f_\zeta d\bar{z})$. Thus, we obtain the essential surjectivity of \mathfrak{G} . The proof of Proposition 2.2 is finished. \blacksquare

As appeared in the proof of Proposition 2.2, we have another equivalent construction of \mathfrak{G} . Let $\mathcal{N}'(V, f)$ denote the cokernel of the endomorphism $\zeta \text{id}_{V \otimes \mathcal{O}_\zeta} - f$ on $V \otimes \mathcal{O}_\zeta$. It naturally induces an \mathcal{O}_{T^\vee} -module $\mathcal{N}(V, f)$. We obtain $\widehat{\text{RFM}}_+(\mathcal{N}(V, f))$, which is naturally isomorphic to $\mathfrak{G}(V, f)$. We obtain a quasi-inverse of \mathfrak{G} as follows. Let E be a semistable bundle of degree 0 on T . We obtain a vector space $H^0(T^\vee, \text{RFM}_-(E))$. If we fix a lift of $\mathcal{S}p(E)$ to $\tilde{\mathcal{S}p}(E) \subset \mathbb{C}$, then the multiplication of ζ induces an endomorphism g_ζ of $H^0(T^\vee, \text{RFM}_-(E))$. The construction of $(H^0(T^\vee, \text{RFM}_-(E)), g_\zeta)$ from E gives a quasi-inverse of \mathfrak{G} .

Let $(E, \bar{\partial}_E)$ be a semistable bundle of degree 0 on T . Let $\tilde{\mathfrak{s}}$ be a subset of \mathbb{C} which is injectively mapped to $T^\vee = \mathbb{C}/L^\vee$.

Corollary 2.6 *We have a unique decomposition $\bar{\partial}_E = \bar{\partial}_{E,0} + f d\bar{z}$ with the following property:*

- $(E, \bar{\partial}_{E,0})$ is holomorphically trivial, i.e., isomorphic to a direct sum of \mathcal{O}_T .
- f is a holomorphic endomorphism of $(E, \bar{\partial}_{E,0})$. We impose the condition that $\mathcal{S}p(H^0(f)) \subset \tilde{\mathfrak{s}}$, where $H^0(f)$ is the induced endomorphism of the space of the global sections of $(E, \bar{\partial}_{E,0})$.

Proof The existence of such a decomposition follows from the essential surjectivity of \mathfrak{G} . Let us show the uniqueness. By considering the spectral decomposition, we have only to consider the case $\tilde{\mathfrak{s}} = \{0\}$. Suppose that $\bar{\partial}_E = \bar{\partial}'_{E,0} + g d\bar{z}$ is another decomposition with the desired property. Because f is holomorphic with respect to $\bar{\partial}_E$, we have $\bar{\partial}'_{E,0}f = 0$ and $[f, g] = 0$ by Lemma 2.5. We put $h = f - g$, which is also nilpotent. The identity induces an isomorphism $(E, \bar{\partial}_{E,0} + h) \simeq (E, \bar{\partial}'_{E,0})$. Because \mathfrak{G} is fully faithful, we obtain $h = 0$. ■

The family version We have a family version of the equivalence. Let S be any complex manifold. Let $\pi_S : T \times S \rightarrow S$ denote the projection. Let $\text{VB}^*(S)$ denote the category of pairs (V, f) of coherent locally free \mathcal{O}_S -module V and its endomorphism f . A morphism $(V, f) \rightarrow (V', f')$ in $\text{VB}^*(S)$ is a morphism of \mathcal{O}_S -modules $g : V \rightarrow V'$ such that $f' \circ g = g \circ f$. Such (V, f) naturally induces an $\mathcal{O}_{\mathbb{C}_\zeta \times S}$ -module $\mathcal{M}(V, f)$. The support is denoted by $\mathcal{S}p(f)$. When we are given a divisor $\tilde{\mathfrak{s}} \subset \mathbb{C}_\zeta \times S$ which is finite and relatively 0-dimensional over S , then $\text{VB}^*(S, \tilde{\mathfrak{s}})$ denote the full subcategory of $(V, f) \in \text{VB}^*(S)$ such that $\mathcal{S}p(f) \subset \tilde{\mathfrak{s}}$.

Let $\text{VB}_0^{ss}(T \times S/S)$ denote the full subcategory of $\mathcal{O}_{T \times S}$ -modules, whose objects are semistable of degree 0 relative to S . When we are given a divisor $\mathfrak{s} \subset T^\vee \times S$ which is relatively 0-dimensional over S , then let $\text{VB}_0^{ss}(T \times S/S, \mathfrak{s})$ denote the full subcategory of $E \in \text{VB}_0^{ss}(T \times S/S)$ such that $\mathcal{S}p(E) \subset \mathfrak{s}$.

Let V be a holomorphic vector bundle on S with a holomorphic endomorphism f . The C^∞ -vector bundle $\pi_S^{-1}V$ is equipped with a naturally induced holomorphic structure obtained as the pull back, denoted by $\bar{\partial}_0$. We obtain a holomorphic vector bundle $\mathfrak{G}(V, f) := (\pi_S^{-1}V, \bar{\partial}_0 + f d\bar{z})$. By Lemma 2.1, \mathfrak{G} gives a functor $\text{VB}^*(S) \rightarrow \text{VB}_0^{ss}(T \times S/S)$. If we are given $\mathfrak{s} \subset T^\vee \times S$ and its lift $\tilde{\mathfrak{s}} \subset \mathbb{C}_\zeta \times S$, it gives an equivalence of the categories $\text{VB}^*(S, \mathfrak{s}) \rightarrow \text{VB}_0^{ss}(T \times S/S, \tilde{\mathfrak{s}})$.

We have another equivalent description of \mathfrak{G} . Let $(V, f) \in \text{VB}^*(S)$. We have the naturally induced $\mathcal{O}_{\mathbb{C}_\zeta \times S}$ -module $\mathcal{M}(V, f)$, which induces an $\mathcal{O}_{T^\vee \times S}$ -module $\mathcal{N}(V, f)$. We have a natural isomorphism $\mathfrak{G}(V, f) \simeq \text{RFM}_-(\mathcal{N}(V, f))$.

If we are given $\mathfrak{s} \subset T^\vee \times S$ with a lift $\tilde{\mathfrak{s}} \subset \mathbb{C}_\zeta \times S$, for an object $E \in \text{VB}_0^{ss}(T \times S/S, \mathfrak{s})$, we obtain a $\mathcal{O}_{\mathbb{C}_\zeta \times S}$ -module $\mathcal{M}(E)$ such that (i) the support of $\mathcal{M}(E)$ is contained in $\tilde{\mathfrak{s}}$, (ii) $\Phi_*\mathcal{M}(E) \simeq \text{RFM}_-(E)$. The multiplication of ζ induces an endomorphism of $\text{RFM}_-(E)$, and hence an endomorphism of $\pi_{S*}(\text{RFM}_-(E))$, denoted by g_ζ , where $\pi_S : T^\vee \times S \rightarrow S$. The construction of $(\pi_{S*} \text{RFM}_-(E), g_\zeta)$ from E gives a quasi-inverse of \mathfrak{G} .

2.1.4 Differential geometric criterion

We recall a differential geometric criterion in terms of the curvature for a metrized holomorphic vector bundle to be semistable of degree 0. Let $(E, \bar{\partial}_E)$ be a holomorphic vector bundle on T with a hermitian metric h . Let $F(h)$ denote the curvature. We use the standard metric $dz d\bar{z}$ of T .

Lemma 2.7 *There exists a constant $\epsilon > 0$, depending only on T and $\text{rank } E$, with the following property:*

- If $|F(h)|_h \leq \epsilon$, then $(E, \bar{\partial}_E, h)$ is semistable of degree 0.

Proof The number $\deg(E) = \int \text{Tr } F(h)$ is an integer. We have $\int |\text{Tr } F(h)| \leq |T| \text{rank } E \epsilon$, where $|T|$ is the volume of T . Hence, we have $\int \text{Tr } F(E_w) = 0$, if ϵ is sufficiently small. For any subbundle $E' \subset E$, by using the decreasing property of the curvature of subbundles, we also obtain $\deg(E') < 1$ and hence $\deg(E') \leq 0$. ■

2.1.5 Small perturbation

We use the metric $dz d\bar{z}$ of T . For any finite dimensional vector space V , let $L_k^p(V)$ be the space of V -valued L_k^p -functions on T , and let $L_k^p(V \otimes \Omega^{i,j})$ be the space of V -valued L_k^p -differential (i, j) -forms. We have the linear map $\int_T : L_k^p(V) \rightarrow V$ given by $\int_T f := |T|^{-1} \int_T f |dz d\bar{z}|$, where $|T|$ denotes the volume of T . The kernel is denoted by $L_k^p(V)_0$. We have a natural inclusion $V \rightarrow L_k^p(V)$ as constant functions. We have the decomposition $L_k^p(V) = L_k^p(V)_0 \oplus V$ as topological vector spaces.

Suppose that V is r -dimensional and equipped with a hermitian metric h_V . Let $p \geq 2$. Let $\mathcal{G}_k^p(V)$ be the space of L_{k+2}^p -maps from T to $\text{GL}(V)$. We set $\mathfrak{A}_k^p(V) := \{\bar{\partial}_0 + A \mid A \in L_{k+1}^p(\text{End}(V) \otimes \Omega^{0,1})\}$, i.e., the space of $(0, 1)$ -type differential operators of the product bundle \underline{V} of the form $\bar{\partial}_0 + A$ ($A \in L_{k+1}^p(\text{End}(V) \otimes \Omega^{0,1})$). We have the natural right \mathcal{G}_k^p -action on $\mathfrak{A}_k^p(V)$ given by $g \bullet \bar{\partial} := g^{-1} \circ \bar{\partial} \circ g = \bar{\partial} + g^{-1} \bar{\partial} g$.

Let Γ be an endomorphism of V which is commutative with its adjoint Γ^\dagger , i.e., it is diagonalizable and, the eigen spaces are orthogonal with respect to h_V . Let $U_1 \subset L_{k+2}^p(\text{End}(V))_0$ be a sufficiently small neighbourhood of 0 such that $1 + U_1 \subset \mathcal{G}_k^p$. Let U_2 be a neighbourhood of 0 in $\text{End}(V)$. We consider the map $\Psi : U_1 \times U_2 \rightarrow \mathfrak{A}_k^p(V)$ given by

$$\Psi(a, b) := (1 + a) \bullet (\bar{\partial}_0 + (\Gamma + b) d\bar{z}).$$

We use the norm on $L_{k+2}^p(\text{End}(V))$ such that $L_{k+2}^p(\text{End}(V)) \simeq L_{k+2}^p(\text{End}(V))_0 \oplus \text{End}(V)$ is an isometry, and the norm on $L_{k+1}^p(\text{End}(V))$ such that $L_{k+2}^p(\text{End}(V)) \rightarrow L_{k+1}^p(\text{End}(V))$, $A \mapsto \bar{\partial}_0 A + \int_T A$ is an isometry.

Proposition 2.8 *Suppose that there exists $\zeta_0 \in \mathbb{C}^*$ such that $Sp(\Gamma)$ is contained in*

$$K_1(L, \zeta_0) := \{\zeta \in \mathbb{C} \mid 0 \leq \text{Im}(\zeta - \zeta_0) \leq (1 - 1/2r)\pi, 0 \leq \text{Im}((\zeta - \zeta_0)\bar{\tau}) \leq (1 - 1/2r)\pi\}.$$

Then, there exist positive constants C_i ($i = 1, 2$), independently from Γ and ζ_0 , such that the following holds:

- For $\mathcal{B} \in L_{k+1}^p(\text{End}(V) \otimes \Omega^{0,1})$ with $|\mathcal{B}| \leq C_1$, there exists a unique $(a, b) \in U_1 \times U_2$ with $|a| + |b| \leq C_2 |\mathcal{B}|$ satisfying $\bar{\partial}_0 + \Gamma d\bar{z} + \mathcal{B} d\bar{z} = \Psi(a, b)$.

Proof We set $K(L) := \{\zeta \in \mathbb{C} \mid |\text{Im}(\zeta)| \leq (1 - 1/2r)\pi, |\text{Im}(\zeta\bar{\tau})| \leq (1 - 1/2r)\pi\}$. We have $Sp(\text{ad}(\Gamma)) \subset K(L)$. In the following, C_i will be positive constants which are independent from Γ and ζ_0 .

We have a morphism $\Phi : L_{k+2}^p(\text{End}(V)) = L_{k+2}^p(\text{End}(V))_0 \oplus \text{End}(V) \rightarrow L_{k+1}^p(\text{End}(V) \otimes \Omega^{0,1})$ given by

$$\Phi(A, B) = \bar{\partial}_0 A + [\Gamma, A] d\bar{z} + B d\bar{z},$$

where $A \in L_{k+2}^p(\text{End}(V))_0$ and $B \in \text{End}(V)$. Similarly, we set $\Phi_0(A, B) = \bar{\partial}_0 A + B d\bar{z}$. By our choice of the norms, Φ_0 is an isometry. We have $|\Phi_0^{-1} \circ \Phi - \text{id}| \leq C_3$ and $|\Phi^{-1} \circ \Phi_0 - \text{id}| \leq C_3$, independently from Γ .

We set $\mathcal{A}(a, b) := \Psi(a, b) - \Psi(0, 0) \in L_{k+1}^p(\text{End}(V) \otimes \Omega^{0,1})$, i.e.,

$$\mathcal{A}(a, b) = (1 + a)^{-1} (\bar{\partial}_0 a + [\Gamma, a]) + \text{Ad}(1 + a) b d\bar{z}.$$

We have $|\mathcal{A}(a, b)| = O(|a| + |b|)$, independently from Γ . The derivative $T_{(a,b)}\Psi$ of Ψ at any $(a, b) \in U_1 \times U_2$ is given by

$$T_{(a,b)}\Psi(X, Y) = \Phi(X, Y) + [\mathcal{A}(a, b), (1 + a)^{-1} X] - [\Psi(0, 0), (1 + a)^{-1} a X] + (\text{Ad}(1 + a) - 1)Y.$$

Hence, we obtain an estimate $|\Phi^{-1} \circ T_{(a,b)}\Psi - \text{id}| \leq C_4(|a| + |b|)$. Then, the claim of Proposition 2.8 follows from the classical inverse function theorem. \blacksquare

Corollary 2.9 *Ψ gives a diffeomorphism of a neighbourhood of $(0, 0)$ in $U_1 \times U_2$ and a neighbourhood of $\bar{\partial}_0 + \Gamma d\bar{z}$ in $\mathfrak{A}_k^p(V)$.* \blacksquare

2.2 Frames

2.2.1 Preliminary

We set $U_1 := \{(x_1, x_2) \mid 0 \leq x_i \leq 1\}$ and $U_2 := \{(\xi_1, \dots, \xi_{n-2}) \mid |\xi_i| \leq 1\}$. Let $T_0 = \mathbb{R}^2/\mathbb{Z}^2$. Let $U_1 \times U_2 \rightarrow T_0 \times U_2$ denote the natural projection. We also use the variables $t_i = x_i$ ($i = 1, 2$) and $t_i = \xi_{i-2}$ ($i = 3, \dots, n$). We also use $x = x_1, y = x_2$.

For a positive integer k , let $S_1(k) := \{(m_1, m_2) \mid m_1 + m_2 = k\}$, and $S_2(k) := \{(m_1, \dots, m_{n-2}) \mid \sum m_i = k\}$. We set $S(k_1, k_2) := S_1(k_1) \times S_2(k_2)$. We put $\partial_{\mathbf{x}}^{\mathbf{m}} := \prod \partial_{x_i}^{m_i}$ and $\partial_{\boldsymbol{\xi}}^{\mathbf{m}} := \prod \partial_{\xi_i}^{m_i}$. We put $N_i(k) := |S_i(k)|$ and $N(k_1, k_2) := N_1(k_1) \times N_2(k_2)$.

Let V be a vector space. For $f \in C^\infty(U_1 \times U_2, V)$, we set

$$D_{\mathbf{x}}^{k_1} D_{\boldsymbol{\xi}}^{k_2} (f) := \left(\partial_{\mathbf{x}}^{\mathbf{m}_1} \partial_{\boldsymbol{\xi}}^{\mathbf{m}_2} f \mid (\mathbf{m}_1, \mathbf{m}_2) \in S(k_1, k_2) \right) \in C^\infty(U_1 \times U_2, V^{N(k_1, k_2)}).$$

Formally, we set $D^0 f := f \in C^\infty(U_1 \times U_2, V)$. We use similar notations for the functions on $T_0 \times U_2$ and $[0, 1] \times U_2$.

2.2.2 Orthonormal frame

Let E be a topologically trivial C^∞ -vector bundle on $T_0 \times U_2$ with a hermitian metric h and a unitary connection ∇ . Let F denote the curvature of ∇ . For a frame \mathbf{v} of E , let $A^{\mathbf{v}} = \sum_{i=1}^n A_i^{\mathbf{v}} dt_i$ denote the connection form of ∇ with respect to \mathbf{v} . We put ${}^1 A^{\mathbf{v}} := A_1^{\mathbf{v}} dt_1 + A_2^{\mathbf{v}} dt_2$ and ${}^2 A^{\mathbf{v}} := \sum_{i=3}^n A_i^{\mathbf{v}} dt_i$. Similarly $F^{\mathbf{v}} = \sum F_{ij}^{\mathbf{v}} dt_i dt_j$ denote the curvature form with respect to \mathbf{v} .

Fix a positive number M . Let ϵ be a small positive number. Assume that $|D_{\mathbf{x}}^{k_1} D_{\boldsymbol{\xi}}^{k_2} F|_h \leq \epsilon$ for any $k_1, k_2 \leq M$.

Lemma 2.10 *If ϵ is sufficiently small, there exist an orthonormal frame \mathbf{v} of (E, h) on $T_0 \times U_2$ and anti-hermitian diagonal matrices $\Lambda^{(x)}, \Lambda^{(y)}$ such that the following holds:*

(A1) $[\Lambda^{(x)}, \Lambda^{(y)}] = 0$.

(A2) Let κ be x or y . For any eigenvalue α of $\Lambda^{(\kappa)}$, we have $0 \leq \alpha < 2\pi$.

(A3) $|{}^1 A^{\mathbf{v}} - \Lambda| \leq C\epsilon^{1/2}$, $|D_{\mathbf{x}}^{k_1} ({}^1 A^{\mathbf{v}})| \leq C\epsilon^{1/2}$, and $|D_{\mathbf{x}}^{k_1} D_{\boldsymbol{\xi}}^{k_2} ({}^1 A^{\mathbf{v}})| \leq C\epsilon$ for any $0 \leq k_1 \leq M$ and $1 \leq k_2 \leq M$, where $\Lambda = \Lambda^{(x)} dx + \Lambda^{(y)} dy$.

(A4) $|D_{\mathbf{x}}^{k_1} D_{\boldsymbol{\xi}}^{k_2} ({}^2 A^{\mathbf{v}})| \leq C\epsilon$ for any $0 \leq k_1, k_2 \leq M$.

Here, the constant C may depend only on $\text{rank } E$ and M .

Proof We shall indicate an outline of the construction, although it is elementary. We say that a quantity \mathcal{P} is $O(\epsilon)$, if $\mathcal{P} \leq C\epsilon$ for some constant C which may depend only on $\text{rank } E$ and M . Let $[a, b]_{\mathbb{Z}}$ denote the set of integers k such that $a \leq k \leq b$. For $j \geq 1$, let H_j be the subset of $U_1 \times U_2$ determined by the condition $t_i = 0$ ($i \in [1, j]_{\mathbb{Z}}$). We set $H_0 := U_1 \times U_2$.

Let \mathbf{u} be an orthonormal frame of $\pi^*(E, h)$ on $U_1 \times U_2$ satisfying $\nabla_{t_i} \mathbf{u} = 0$ on H_{i-1} for any i . We have $A_p^{\mathbf{u}} = 0$ on H_{p-1} by the construction. For $j < p$, we have $\partial_{t_j} A_p^{\mathbf{u}} = F_{jp}^{\mathbf{u}}$ on H_{j-1} . For a monomial P of $\partial_{t_{j+1}}, \dots, \partial_{t_n}$, we have $\partial_{t_j}^{\alpha+1} P A_p^{\mathbf{u}} = \partial_{t_j}^{\alpha} P F_{jp}^{\mathbf{u}}$ on H_{j-1} . Hence, for $j \leq p$ and for a monomial $\tilde{P} = \prod_{i=j}^n \partial_{t_i}^{m_i}$ satisfying $m_1 + m_2 \leq M$ and $\sum_{i>2} m_i \leq M$, we obtain $\tilde{P} A_p^{\mathbf{u}} = O(\epsilon)$ on H_{j-1} by a descending induction. In particular, we obtain $D_{\mathbf{x}}^{k_1} D_{\boldsymbol{\xi}}^{k_2} A_p^{\mathbf{u}} = O(\epsilon)$ for any $(k_1, k_2) \in [0, M]_{\mathbb{Z}}^2$.

Let $G^{(x)} : H_1 \rightarrow U(\text{rank } E)$ be determined by $\mathbf{u}_{|(1, y, \boldsymbol{\xi})} = \mathbf{u}_{|(0, y, \boldsymbol{\xi})} G^{(x)}(y, \boldsymbol{\xi})$. By the equation

$$\partial_{t_i} G^{(x)}(t_2, \dots, t_n) - G^{(x)}(t_2, \dots, t_n) A_{i|(1, t_2, \dots, t_n)}^{\mathbf{u}} + A_{i|(0, t_2, \dots, t_n)}^{\mathbf{u}} G^{(x)}(t_2, \dots, t_n) = 0,$$

we obtain $|D_{\mathbf{x}}^1 G^{(x)}| + |D_{\boldsymbol{\xi}}^1 G^{(x)}| = O(\epsilon)$. By an easy induction, we obtain $|D_{\mathbf{x}}^{k_1} D_{\boldsymbol{\xi}}^{k_2} G^{(x)}| = O(\epsilon)$ for any $(k_1, k_2) \in [0, M]_{\mathbb{Z}}^2 \setminus \{(0, 0)\}$. We also have $|G^{(x)}(y, \boldsymbol{\xi}) - G^{(x)}(y', \boldsymbol{\xi}')| = O(\epsilon)$.

Let $G^{(y)}(x, \xi)$ be determined by $\mathbf{u}|_{(x,1,\xi)} = \mathbf{u}|_{(x,0,\xi)} G^{(y)}(x, \xi)$. Similarly, we have $|D_{\mathbf{x}}^{k_1} D_{\xi}^{k_2} G^{(y)}| = O(\epsilon)$ for any $(k_1, k_2) \in [0, M]_{\mathbb{Z}}^2 \setminus \{(0, 0)\}$, and $|G^{(y)}(x, \xi) - G^{(y)}(x', \xi')| = O(\epsilon)$. Because $G^{(y)}(0, \xi)G^{(x)}(1, \xi) = G^{(x)}(0, \xi)G^{(y)}(1, \xi)$, we have $[G^{(y)}(0, 0), G^{(x)}(0, 0)] = O(\epsilon)$.

By applying Lemma 2.20 below to $G^{(x)}(0, 0)$ and $G^{(y)}(0, 0)$, we obtain $\tilde{G}^{(x)}$ and $\tilde{G}^{(y)}$ such that $[\tilde{G}^{(x)}, \tilde{G}^{(y)}] = 0$ and $|\tilde{G}^{(x)} - G^{(x)}(0, 0)| = O(\epsilon^{1/2})$ and $|\tilde{G}^{(y)} - G^{(y)}(0, 0)| = O(\epsilon^{1/2})$.

We take $\Lambda^{(x)}, \Lambda^{(y)} \in \mathfrak{u}(\text{rank } E)$ such that $\exp(\Lambda^{(x)}) = \tilde{G}^{(x)}$, $\exp(\Lambda^{(y)}) = \tilde{G}^{(y)}$ and that **(A1)** and **(A2)** are satisfied. We put $g^{(x)}(x) := \exp(-x \Lambda^{(x)})$, $g^{(y)}(y) := \exp(-y \Lambda^{(y)})$, and $g(x, y) := g^{(x)}(x) g^{(y)}(y)$. We obtain an orthonormal frame $\mathbf{u}' := \mathbf{u} g(x, y)$ of $\pi^*(E, h)$. Let $A' := A^{\mathbf{u}'}$. We have $|A' - \Lambda| = O(\epsilon)$ and $|D_{\mathbf{x}}^{k_1} D_{\xi}^{k_2} A'| = O(\epsilon)$ for any $(k_1, k_2) \in [0, M]_{\mathbb{Z}}^2 \setminus \{(0, 0)\}$.

Let $G'^{(x)}(y, \xi)$ and $G'^{(y)}(x, \xi)$ be determined by

$$\mathbf{u}'|_{(1,y,\xi)} = \mathbf{u}'|_{(0,y,\xi)} G'^{(x)}(y, \xi), \quad \mathbf{u}'|_{(x,1,\xi)} = \mathbf{u}'|_{(x,0,\xi)} G'^{(y)}(x, \xi).$$

We have $G'^{(x)}(y, \xi) = g^{(y)}(y)^{-1} G^{(x)}(y, \xi) (\tilde{G}^{(x)})^{-1} g^{(y)}(y)$ and hence $|G'^{(x)} - 1| = O(\epsilon^{1/2})$. We have

$$dG'^{(x)} = g^{(y)}(y)^{-1} dG^{(x)}(y, \xi) (\tilde{G}^{(x)})^{-1} g^{(y)}(y) - [g^{(y)}(y)^{-1} dg^{(y)}(y), (G'^{(x)} - 1)].$$

Hence, we have $|D_y^1 G'^{(x)}| = O(\epsilon^{1/2})$ and $|D_{\xi}^1 G'^{(x)}| = O(\epsilon)$. By an easy induction, we obtain $|D_y^{k_1} G'^{(x)}| = O(\epsilon^{1/2})$ and $|D_y^{k_1} D_{\xi}^{k_2} G'^{(x)}| = O(\epsilon)$ for $(k_1, k_2) \in [0, M]_{\mathbb{Z}} \times [1, M]_{\mathbb{Z}}$. We have similar estimates for $G'^{(y)}$.

Let $\chi(x)$ be a non-negative valued C^∞ -function on $[0, 1]$ such that $\chi(x) = 0$ ($x \leq 1/3$) and $\chi(x) = 1$ ($x \geq 2/3$). We put $h_2(x, y, \xi) := \chi(x) \exp^{-1}(G'^{(x)}(y, \xi))$. By construction, we have $|D_{\mathbf{x}}^{k_1} h_2| = O(\epsilon^{1/2})$ for $k_1 \in [0, M]_{\mathbb{Z}}$, and $|D_{\mathbf{x}}^{k_1} D_{\xi}^{k_2} h_2| = O(\epsilon)$ for $(k_1, k_2) \in [0, M]_{\mathbb{Z}} \times [1, M]_{\mathbb{Z}}$.

Let $g_2 := \exp(h_2)$, and we set $\mathbf{u}'' := \mathbf{u}' g_2$. Let $A'' = A^{\mathbf{u}''}$. We have $A'' = g_2^{-1} A' g_2 + g_2^{-1} dg_2$. Hence, we have $|A'' - \Lambda| = O(\epsilon^{1/2})$, $|D_{\mathbf{x}}^{k_1} (A'')| = O(\epsilon^{1/2})$ and $|D_{\mathbf{x}}^{k_1} D_{\xi}^{k_2} (A'')| = O(\epsilon)$ for $(k_1, k_2) \in [0, M]_{\mathbb{Z}} \times [1, M]_{\mathbb{Z}}$. We also have $|D_{\mathbf{x}}^{k_1} D_{\xi}^{k_2} (2A'')| = O(\epsilon)$ for $(k_1, k_2) \in [0, M]_{\mathbb{Z}}^2$.

We put $G''(x, \xi) := g_2(x, 0, \xi)^{-1} G'(x, \xi) g_2(x, 1, \xi)$. We have $G''(x, \xi) \mathbf{u}''|_{(x,1,\xi)} = \mathbf{u}''|_{(x,0,\xi)} G''(x, \xi)$. We have $|G''(x, \xi) - 1| = O(\epsilon^{1/2})$, $|D_{\mathbf{x}}^{k_1} G''| = O(\epsilon^{1/2})$ and $|D_{\mathbf{x}}^{k_1} D_{\xi}^{k_2} G''| = O(\epsilon)$ for $(k_1, k_2) \in [0, M]_{\mathbb{Z}} \times [1, M]_{\mathbb{Z}}$.

We put $g_3 := \exp(\chi(y) \exp^{-1}(G''(x, \xi)))$, and $\mathbf{v} := \mathbf{u}'' g_3$. Then, it naturally gives an orthonormal frame of (E, h) on $T_0 \times U_2$. By construction, we have the desired estimate for the connection form $A^{\mathbf{v}}$. Thus, the proof of Lemma 2.10 is finished. \blacksquare

2.2.3 Partially almost holomorphic frame

We identify the C^∞ -manifolds $T_0 := \mathbb{C}^2/\mathbb{Z}^2$ and T by the diffeomorphism $T_0 \simeq T$ given by $(x, y) \mapsto x + \tau y = z$. We have the description $\Lambda = \Gamma d\bar{z} - \bar{\Gamma} dz$, where Λ is as in Lemma 2.10. Let $\nabla_{\bar{z}} := \nabla(\partial_{\bar{z}})$ and $\nabla_z := \nabla(\partial_z)$. For a frame \mathbf{w} , let $A_z^{\mathbf{w}}$ and $A_{\bar{z}}^{\mathbf{w}}$ be determined by $\nabla_z \mathbf{w} = \mathbf{w} A_z^{\mathbf{w}}$ and $\nabla_{\bar{z}} \mathbf{w} = \mathbf{w} A_{\bar{z}}^{\mathbf{w}}$, respectively. Let $H(h, \mathbf{w})$ denote a function from $T \times U_2$ to the space of $\text{rank}(E)$ -th positive definite hermitian matrices, whose (i, j) -entries are $h(w_i, w_j)$. When a function f on $T \times U_2$ is regarded as a function $\tilde{f} : U_2 \rightarrow L_k^p(T)$, we obtain an $\mathbb{R}_{\geq 0}$ -valued function $\|f\|_{L_k^p(\xi)} := \|\tilde{f}(\xi)\|_{L_k^p(T)}$ on U_2 .

Proposition 2.11 *If $\epsilon > 0$ is sufficiently small, there exists a frame \mathbf{u} of E on $T \times U_2$ with the following property:*

- $A_{\bar{z}}^{\mathbf{u}}$ is constant along the T -direction, and $|A_{\bar{z}}^{\mathbf{u}} - \Gamma| = O(\epsilon^{1/2})$.
- $\|A_z^{\mathbf{u}} + \bar{\Gamma}\|_{L_M^p} = O(\epsilon^{1/2})$ and $\|D_{\xi}^k A_z^{\mathbf{u}}\|_{L_M^p} = O(\epsilon)$ for $k \in [1, M]_{\mathbb{Z}}$.
- $\|D_{\xi}^k (2A^{\mathbf{u}})\|_{L_M^p} = O(\epsilon)$ for $k \in [0, M]_{\mathbb{Z}}$.

Moreover, $\|H(h, \mathbf{u}) - I\|_{L_{M+1}^p} = O(\epsilon^{1/2})$ and $\|D_{\xi}^k H(h, \mathbf{u})\|_{L_{M+1}^p} = O(\epsilon)$ for $k \in [1, M]_{\mathbb{Z}}$.

Proof Let \mathbf{v} be as in Lemma 2.10. We have $\nabla_{\bar{z}}\mathbf{v} = \mathbf{v}(\Gamma + N) d\bar{z}$, where $\|N\|_{L_M^p} = O(\epsilon^{1/2})$ and $\|D_{\xi}^k N\|_{L_M^p} = O(\epsilon)$. By Proposition 2.8, there exist functions $a : U_2 \rightarrow L_{M+1}^p(M_{\text{rank } E}(\mathbb{C}))_0$ and $b : U_2 \rightarrow M_{\text{rank } E}(\mathbb{C})$ satisfying the following:

- $\|a\|_{L_{M+1}^p} = O(\epsilon^{1/2})$ and $\|D_{\xi}^k a\|_{L_{M+1}^p} = O(\epsilon)$ for $k \in [1, M]_{\mathbb{Z}}$.
- $|b| = O(\epsilon^{1/2})$ and $|D_{\xi}^k b| = O(\epsilon)$ for $k \in [1, M]$.
- $(1+a) \bullet (\nabla_{\bar{z},0} + (\Gamma + b) d\bar{z}) = \nabla_{\bar{z}}$, where $\nabla_{\bar{z},0}$ is given by $\nabla_{\bar{z},0}\mathbf{v} = 0$.

Let $\mathbf{u} := \mathbf{v}(1+a)$. By construction, we have $\nabla_{\bar{z}}\mathbf{u} = \mathbf{u}(\Gamma + b) d\bar{z}$. The other estimates for $A_{\bar{z}}^{\mathbf{u}}$ and ${}^2A^{\mathbf{u}}$ are also satisfied. Because $H(h, \mathbf{u}) = {}^t(1+a)\overline{(1+a)}$, we obtain the estimate for $H(h, \mathbf{u})$. \blacksquare

Remark 2.12 If $A_{\bar{z}}^{\mathbf{w}}$ is constant along the T -direction, such a frame \mathbf{w} is called a *partially almost holomorphic frame*, in this paper. \blacksquare

2.2.4 Spectra

Let E_{ξ} denote the holomorphic bundle on T given by $E|_{T \times \xi}$ with $\nabla_{\bar{z}|T \times \xi}$. According to Lemma 2.7, if ϵ is sufficiently small, E_{ξ} are semistable of degree 0 for any $\xi \in U_2$. We have the spectrum $Sp(E_{\xi}) \subset T^{\vee}$. We regard it as a point in $\text{Sym}^r T^{\vee}$, where $r := \text{rank } E$. The point is denoted by $[Sp(E_{\xi})]$. Let Γ be as in §2.2.3. The eigenvalues of Γ gives a point in $\text{Sym}^r \mathbb{C}$, denoted by $[Sp(\Gamma)]$. The quotient map $\Phi : \mathbb{C} \rightarrow T^{\vee}$ induces $\text{Sym}^r \mathbb{C} \rightarrow \text{Sym}^r T^{\vee}$, denoted by Φ . Recall that $\text{Sym}^r T^{\vee}$ is naturally a smooth complex manifold. Let $d_{\text{Sym}^r T^{\vee}}$ be a distance induced by a C^{∞} -Riemannian metric.

Corollary 2.13 *There exist $\epsilon_0 > 0$ and $C > 0$ such that the following holds if $\epsilon \leq \epsilon_0$:*

$$d_{\text{Sym}^r T^{\vee}}([Sp(E_{\xi})], [\Phi([Sp(\Gamma)])]) \leq C\epsilon^{1/2}$$

Proof Let \mathbf{u} be a frame as in Proposition 2.11. Recall that $\text{Sym}^r \mathbb{C}$ is naturally a complex manifold. We take a distance $d_{\text{Sym}^r \mathbb{C}}$ induced by a C^{∞} -Riemannian metric. We have $d_{\text{Sym}^r \mathbb{C}}([Sp(\Gamma)], [Sp(A_{\bar{z}}^{\mathbf{u}})]) \leq C_1\epsilon^{1/2}$. There exists $\zeta_0 \in \mathbb{C}$ such that $Sp(\Gamma)$ and $Sp(A_{\bar{z}}^{\mathbf{u}})$ are contained in $K_1(L, \zeta_0)$. Note that the restriction of Φ to $\text{Sym}^r K_1(L, \zeta_0)$ is Lipschitz continuous, and the Lipschitz constant is uniform for ζ_0 . Then, the claim of the corollary follows. \blacksquare

2.3 Estimates

2.3.1 Preliminary

We continue to use the setting in §2.2. Let $r := \text{rank } E$. We impose additional assumptions.

Assumption 2.14

- ϵ is sufficiently small so that E_{ξ} is semistable of degree 0 for any $\xi \in U_2$.
- We are given a point $\mathfrak{k} \in \text{Sym}^r \mathbb{C}$ such that (i) \mathfrak{k} is contained in $\text{Sym}^r K_1(L, \zeta_0)$ for some $\zeta_0 \in \mathbb{C}$, where $K_1(L, \zeta_0)$ is as in §2.1.5, (ii) $d_{\text{Sym}^r T^{\vee}}([Sp(E_{\xi})], \Phi(\mathfrak{k})) \leq C\epsilon^{1/2}$, where $\Phi : \text{Sym}^r \mathbb{C} \rightarrow \text{Sym}^r T^{\vee}$ is induced by $\mathbb{C} \rightarrow T^{\vee}$. \blacksquare

Let $Z \subset \mathbb{C}$ be the minimum of the subset $Z' \subset K_1(L, \zeta_0)$ such that $\mathfrak{k} \in \text{Sym}^r Z'$. For $\nu \in Z$, let $\mathfrak{m}(\nu)$ denote the multiplicity of ν in \mathfrak{k} . Let Γ_0 be the diagonal matrix $\bigoplus_{\nu \in Z} \nu I_{\mathfrak{m}(\nu)}$. Let Λ be as in Lemma 2.10. By Corollary 2.13, we may assume $\Lambda = \Gamma_0 d\bar{z} - \bar{\Gamma}_0 dz$, under $z = x + \tau y$.

We have the spectral decomposition $E_{\xi} = \bigoplus_{\nu' \in T^{\vee}} E_{\xi, \nu'}$. Let $E_{\nu, \xi}$ be the direct sum of $E_{\xi, \nu'}$, where ν' is contained in $\epsilon^{1/2}$ -neighbourhood of ν . We obtain a decomposition $E_{\xi} = \bigoplus_{\nu \in Z} E_{\nu, \xi}$. It induces a C^{∞} -decomposition $E = \bigoplus_{\nu \in Z} E_{\nu}$, which is compatible with $\nabla_{\bar{z}}$. We may assume that the partially almost holomorphic frame \mathbf{u} in Proposition 2.11 is compatible with the decomposition.

We have the decomposition $\nabla_{\bar{z}} = \nabla_{\bar{z},0} + f$ such that (i) $(E, \nabla_{\bar{z},0})|_{T \times \{\xi\}}$ are holomorphically trivial, (ii) $\nabla_{\bar{z}}(f) = 0$, (iii) $d_{\text{Sym}^r \mathbb{C}}([\mathcal{S}p(f)], \mathfrak{k}) = O(\epsilon^{1/2})$. For each $\xi \in U_2$, we obtain the vector space \mathcal{V}_ξ of the holomorphic global sections of $(E, \nabla_{\bar{z},0})|_{T \times \{\xi\}}$. It is easy to see that \mathcal{V}_ξ ($\xi \in U_2$) naturally gives a C^∞ -vector bundle \mathcal{V} , and that we have a natural isomorphism $p^*\mathcal{V} \simeq E$ as C^∞ -bundles. We identify them by the isomorphism. A C^∞ -section s of $p^*\mathcal{V}$ is constant along the T -direction, if and only if $\nabla_{\bar{z},0}s = 0$ under the identification. It can be regarded as a section of \mathcal{V} . We have the decomposition $\mathcal{V} = \bigoplus_{\nu \in Z} \mathcal{V}_\nu$, corresponding to $E = \bigoplus_{\nu \in Z} E_\nu$.

2.3.2 Space of functions

Let $C_\xi^M L_{M,\mathbf{x}}^p$ denote the space of C^M -functions $U_2 \rightarrow L_M^p(T)$. Let $C_\xi^M L_{M,\mathbf{x}}^p(E)$ denote the sections $f = \sum f_i u_i$ of E such that $f_i \in C_\xi^M L_{M,\mathbf{x}}^p$, where $\mathbf{u} = (u_i)$ is a frame as in Proposition 2.11. It is independent of the choice of \mathbf{u} . We have the naturally defined integration $\int_T : C_\xi^M L_{M,\mathbf{x}}^p(E) \rightarrow C^M(U_2, \mathcal{V})$. The kernel is denoted by $C_\xi^M L_{M,\mathbf{x}}^p(E)_0$. Similar spaces are defined for $\text{End}(E)$ and $\text{Hom}(E_i, E_j)$. We set

$$C_\xi^M L_{M,\mathbf{x}}^p(\text{End}(E))^\circ := \bigoplus_{\nu} C^M(U_2, \text{End}(\mathcal{V}_\nu))$$

$$C_\xi^M L_{M,\mathbf{x}}^p(\text{End}(E))^\perp := \bigoplus_{\nu} C_\xi^M L_{M,\mathbf{x}}^p(\text{End}(E_\nu))_0 \oplus \bigoplus_{\nu \neq \mu} C_\xi^M L_{M,\mathbf{x}}^p(\text{Hom}(E_\nu, E_\mu))$$

We have a decomposition $C_\xi^M L_{M,\mathbf{x}}^p(\text{End}(E)) = C_\xi^M L_{M,\mathbf{x}}^p(\text{End}(E))^\circ \oplus C_\xi^M L_{M,\mathbf{x}}^p(\text{End}(E))^\perp$. For any $s \in C_\xi^M L_{M,\mathbf{x}}^p(\text{End}(E))$, the corresponding decomposition is denoted by $s = s^\circ + s^\perp$. We use similar notations for sections of $\text{End}(E) \otimes \Omega_T^{i,j}$.

2.3.3 Some estimates

Let \mathbf{u} be a frame as in Proposition 2.11. We set $H(h, \mathbf{u})_{i,j} := h(u_i, u_j)$, and we obtain a function $H(h, \mathbf{u})$ from $T \times U_2$ to the space \mathcal{H} of positive definite hermitian r -th matrices. Each entry is $C_\xi^M L_{M,\mathbf{x}}^p$ -class. Let H_1 be a function of U_2 to \mathcal{H} determined by $(H_1)^2 = \int_T H(h, \mathbf{u})$. Then, we have $|H_1 - I| = O(\epsilon^{1/2})$ and $|D_\xi^k H_1| = O(\epsilon)$ for $k \in [1, M]_\mathbb{Z}$. Note that $\mathbf{u}' := \mathbf{u} H_1$ also has the property in Proposition 2.11. So, we may assume that $\int_T H(h, \mathbf{u}) = I$ from the beginning.

We set $\tilde{g} := \overline{H(h, \mathbf{u})}$. We have $\|\tilde{g} - I\|_{L_{M+1}^p} = O(\epsilon^{1/2})$, $\|D_\xi^k \tilde{g}\|_{L_{M+1}^p} = O(\epsilon)$ ($k \in [1, M]_\mathbb{Z}$), and $\int_T \tilde{g} = I$.

Lemma 2.15 *There exist $C > 0$ and $\epsilon_0 > 0$, such that $\|\tilde{g} - I\|_{L_{M+2}^p} \leq C \|F_{z\bar{z}}^\perp\|_{L_M^p}$ holds if $\epsilon < \epsilon_0$. In particular, $\sup_{T \times \{\xi\}} |\tilde{g} - I| \leq C' \|F_{z\bar{z}}^\perp\|_{L^2}$ for some $C' > 0$.*

Proof We put $B := A_z^\mathbf{u}$. Then, we have $A_z^\mathbf{u} = -\tilde{g}^{-1}({}^t\bar{B})\tilde{g} + \tilde{g}^{-1}\partial_z\tilde{g}$. Let $\mathcal{B}_{z\bar{z}}$ be the matrix-valued function determined by $F_{z\bar{z}}\mathbf{u} = \mathbf{u} \mathcal{B}_{z\bar{z}}$. We have $\mathcal{B}_{z\bar{z}} = \partial_z A_z^\mathbf{u} - \partial_{\bar{z}} A_z^\mathbf{u} + [A_z^\mathbf{u}, A_z^\mathbf{u}]$. Hence, we have the following equation:

$$\mathcal{B}_{z\bar{z}} = [\tilde{g}^{-1}({}^t\bar{B})\tilde{g}, \tilde{g}^{-1}\partial_z\tilde{g}] - \tilde{g}^{-1}\partial_{\bar{z}}\partial_z(\tilde{g}) + (\tilde{g}^{-1}\partial_{\bar{z}}\tilde{g})(\tilde{g}^{-1}\partial_z\tilde{g}) - [\tilde{g}^{-1}({}^t\bar{B})\tilde{g}, B] - [B, \tilde{g}^{-1}\partial_z\tilde{g}] \quad (2)$$

Let $b := \tilde{g} - I$. We have a polynomial $Q(t_1, t_2, t_3)$ without constant and linear terms, whose coefficients depend on Γ , such that

$$(\partial_{\bar{z}} + \text{ad}(B)) \circ (\partial_z - \text{ad}({}^t\bar{B}))b = -\tilde{g}\mathcal{B}_{z\bar{z}} - [{}^t\bar{B}, B] + Q(b, \partial_z b, \partial_{\bar{z}} b). \quad (3)$$

By taking the \perp -part, we obtain the following

$$(\partial_{\bar{z}} + \text{ad}(B)) \circ (\partial_z - \text{ad}({}^t\bar{B}))b = -\tilde{g}\mathcal{B}_{z\bar{z}}^\perp + Q(b, \partial_z b, \partial_{\bar{z}} b)^\perp. \quad (4)$$

We obtain

$$(\|b\|_{L_{m+2}^p} + \|\partial_z b\|_{L_{m+1}^p} + \|\partial_{\bar{z}} b\|_{L_{m+1}^p}) \leq C_2 \|F_{z\bar{z}}^\perp\|_{L_m^p} + C_2 \epsilon^{1/2} (\|b\|_{L_{m+2}^p} + \|\partial_z b\|_{L_{m+1}^p} + \|\partial_{\bar{z}} b\|_{L_{m+1}^p})$$

Hence, we obtain $\|b\|_{L_{m+2}^p} \leq C_3 \|F_{z\bar{z}}^\perp\|_{L_m^p}$. ■

Lemma 2.16 Let a_1 and a_2 be sections of $\text{End}(E)_{|T \times \{\xi\}}$. Assume that $a_1 = a_1^\perp$ and $a_2 = a_2^\circ$. Then, we have

$$\left| \int_T h(a_1, a_2) \right| \leq \|a_1\|_{L^2} \|a_2\|_{L^2} \|(F_{z\bar{z}}^\perp)_{|T \times \{\xi\}}\|_{L^2}.$$

Proof It follows from Lemma 2.15 and $\overline{H(h, \mathbf{u})} = \tilde{g}$. ■

Lemma 2.17 Let \mathcal{P} be an endomorphism of E , and let \mathcal{P}^\dagger denote the adjoint with respect to h . Let \mathcal{R} (resp. \mathcal{R}^\dagger) be the matrix representing \mathcal{P} (resp. \mathcal{P}^\dagger) with respect to \mathbf{u} . Then, we have

$$\begin{aligned} (\mathcal{R}^\dagger)^\circ &= (\overline{\mathcal{R}})^\circ + O(|\mathcal{R}^\perp| \|F_{z\bar{z}}^\perp\|_{L^2}) + O(\|F_{z\bar{z}}^\perp\|_{L^2}^2 |\overline{\mathcal{R}}^\circ|) \\ (\mathcal{R}^\dagger)^\perp &= (\overline{\mathcal{R}})^\perp + O(|\mathcal{R}^\perp| \|F_{z\bar{z}}^\perp\|_{L^2}) + O(\|F_{z\bar{z}}^\perp\|_{L^2} |\mathcal{R}^\circ|) \end{aligned}$$

In particular, we have $|(\mathcal{R}^\dagger)^\perp| = |\mathcal{R}^\perp| + O(|\mathcal{R}| \|F_{z\bar{z}}^\perp\|_{L^2})$.

Proof Let $H = H(h, \mathbf{u})$. We have $\mathcal{R}^\dagger = \overline{H^{-1}(\overline{\mathcal{R}})} H$. Then, the claim follows from the estimate for H . ■

Lemma 2.18 For $k \in [1, M]_{\mathbb{Z}}$, we have $\|D_{\xi}^k \tilde{g}\|_{L_{m+2}^p} = O\left(\sum_{j=0}^k \|D_{\xi}^j F_{z\bar{z}}^\perp\|_{L_m^p}\right)$. ■

Proof We obtain the estimate from (4) by a standard inductive argument. ■

2.4 Modification to a commutative pair (Appendix)

Let (X, d) be a metric space. Let $S \subset X$ be a finite subset. We fix $\epsilon_0 > 0$.

Lemma 2.19 There exist a non-negative integer $N \leq |S|$ and a decomposition $S = \coprod_{j \in \Lambda} \tilde{S}_j$ with the following property:

- If $j \neq k$, we have $d(P, Q) > 90(100|S|)^N \epsilon_0$ for any $P \in \tilde{S}_j$ and $Q \in \tilde{S}_k$.
- We have $d(P, Q) \leq 4(100|S|)^N \epsilon_0$ for any $P, Q \in \tilde{S}_j$.

Proof We make general preparations. In general, for a given finite graph Γ , let $V(\Gamma)$ denote the set of vertexes, and let $C(\Gamma)$ denote the set of connected components of the geometric realization of Γ . We have the decomposition $\Gamma = \coprod_{j \in C(\Gamma)} \Gamma_j$ into the connected components. We put $m(\Gamma) := \max\{|V(\Gamma_j)| \mid j \in C(\Gamma)\}$.

For any positive number δ and any finite subset $T \subset X$, we have a unique graph $\Gamma(T, \delta)$ with a bijection $\iota : V(\Gamma(T, \delta)) \simeq T$ determined by the condition.

- $P, Q \in V(\Gamma(T, \delta))$ are connected by an edge if and only if $d(\iota(P), \iota(Q)) \leq \delta$ and $P \neq Q$.

For such a graph, we will not distinguish $V(\Gamma(T, \delta))$ and T .

Let us construct a decomposition as in the claim of the proposition. For $j = 0, 1, \dots, N$, we shall inductively construct finite subsets $S^{(j)} \subset X$ and graphs $\Gamma^{(j)}$ such that $V(\Gamma^{(j)}) = S^{(j)}$, until $m(\Gamma^{(N)}) = 1$.

We set $S^{(0)} := S$ and $\Gamma^{(0)} := \Gamma(S^{(0)}, 100\epsilon_0)$. Suppose that we have already constructed $(S^{(j)}, \Gamma^{(j)})$ for $j = 0, \dots, \ell$ with $m(\Gamma^{(j)}) > 1$ ($j < \ell$). If $m(\Gamma^{(\ell)}) = 1$, we stop here. Let us consider the case $m(\Gamma^{(\ell)}) > 1$. We have the decomposition $S^{(\ell)} = \coprod_{j \in C(\Gamma^{(\ell)})} S_j^{(\ell)}$ according to the decomposition of the graph $\Gamma^{(j)}$ into the connected components. For each $j \in C(\Gamma^{(\ell)})$, we choose a point $P_j^{(\ell)} \in S_j^{(\ell)}$. Then, we define

$$S^{(\ell+1)} := \{P_j^{(\ell)} \mid j \in C(\Gamma^{(\ell)})\}, \quad \Gamma^{(\ell+1)} := \Gamma(S^{(\ell+1)}, 100(100|S|)^\ell \epsilon_0).$$

The inductive procedure finishes at some ℓ . By the construction, we have a naturally defined map $\pi_i : S^{(i)} \rightarrow S^{(i+1)}$ such that $\pi_i(P) = P_j^{(i)} \in S^{(i+1)}$ for $P \in S_j^{(i)}$. They induce a map $\pi : S \rightarrow S^{(\ell)}$. For $R \in S^{(\ell)}$, we set $\tilde{S}_R := \pi^{-1}(R)$. By the construction, if P is contained in \tilde{S}_R , we have

$$d(P, R) \leq (100|S|)^{\ell-1} \epsilon_0 + (100|S|)^{\ell-2} \epsilon_0 + \dots + (100|S|) \epsilon_0 \leq 2(100|S|)^{\ell-1} \epsilon_0$$

Hence, for $P_1, P_2 \in \tilde{S}_R$, we have $d(P_1, P_2) \leq 4(100|S|)^{\ell-1}\epsilon_0$. If $P_i \in \tilde{S}_{R_i}$ ($i = 1, 2$) with $R_1 \neq R_2$, we have

$$d(P_1, P_2) \geq d(R_1, R_2) - d(P_1, R_1) - d(P_2, R_2) \geq 100(100|S|)^{\ell-1}\epsilon_0 - 8(100|S|)^{\ell-1}\epsilon_0 \geq 90(100|S|)^{\ell-1}\epsilon_0.$$

Hence, the decomposition $S = \coprod_{R \in S^{(\ell)}} \tilde{S}_R$ has the desired property. \blacksquare

Let $U(r)$ denote the r -th unitary group.

Lemma 2.20 *There exist positive constants ϵ_1 and C_1 , such that the following holds for any $0 < \epsilon < \epsilon_1$.*

- For any $G^{(i)} \in U(r)$ ($i = 1, 2$) such that $[[G^{(1)}, G^{(2)}]] \leq \epsilon$, there exist $\tilde{G}^{(i)} \in U(r)$ ($i = 1, 2$) satisfying $[\tilde{G}^{(1)}, \tilde{G}^{(2)}] = 0$ and $|\tilde{G}^{(i)} - G^{(i)}| \leq C_1 \epsilon^{1/2}$.

Proof In the following, C_i are positive constants which depend only on r . We may assume that $G^{(1)}$ is diagonal, $G^{(1)} = \bigoplus_{\alpha \in Sp(G^{(1)})} \alpha I_\alpha$, with the eigen decomposition $\mathbb{C}^r = \bigoplus_{\alpha \in Sp(G^{(1)})} V_\alpha$. Applying Lemma 2.19 to the set $Sp(G^{(1)})$ with $\epsilon_0 = \epsilon^{1/2}$, we have a decomposition $Sp(G^{(1)}) = \coprod_{k \in \Lambda} \tilde{S}_k$ and a non-negative integer $\ell \leq r$ with the following property:

- We have $d(\alpha_1, \alpha_2) > 90(100r)^\ell \epsilon^{1/2}$ for $\alpha_i \in \tilde{S}_{k_i}$ with $k_1 \neq k_2$.
- We have $d(\alpha, \beta) < 4(100r)^\ell \epsilon^{1/2}$ for $\alpha, \beta \in \tilde{S}_k$.

We obtain a decomposition $\mathbb{C}^r = \bigoplus_{k \in \Lambda} V_k$, where $V_k = \bigoplus_{\alpha \in \tilde{S}_k} V_\alpha$. For each $k \in \Lambda$, we choose $\beta_k \in \tilde{S}_k$. Let $\tilde{G}^{(1)}$ be $\bigoplus_{k \in \Lambda} \beta_k \left(\bigoplus_{\alpha \in \tilde{S}_k} I_\alpha \right)$. By construction, we have $|\tilde{G}^{(1)} - G^{(1)}| \leq C_2 \epsilon^{1/2}$.

We have the decompositions $G^{(1)} = \sum_{k \in \Lambda} G_{k,k}^{(1)}$ and $G^{(2)} = \sum_{k,m \in \Lambda} G_{k,m}^{(2)}$, according to $V = \bigoplus_{k \in \Lambda} V_k$. Because $|G^{(1)} G^{(2)} - G^{(2)} G^{(1)}| \leq \epsilon$, we have $|G_{k,k}^{(1)} G_{k,m}^{(2)} - G_{k,m}^{(2)} G_{m,m}^{(1)}| \leq C_3 \epsilon$. Because the difference of the eigenvalues of $G_{k,k}^{(1)}$ and $G_{m,m}^{(1)}$ is dominated by $C_4 \epsilon^{1/2}$ from below, we obtain $|G_{k,m}^{(2)}| \leq C_5 \epsilon^{1/2}$. We also obtain $|G_{k,k}^{(2)} - \overline{G_{k,k}^{(2)}}| \leq C_6 \epsilon^{1/2}$. Hence, we can find $\tilde{G}_{k,k}^{(2)'} \in U(\text{rank } V_k)$ such that $|\tilde{G}_{k,k}^{(2)'} - G_{k,k}^{(2)}| \leq C_7 \epsilon^{1/2}$. We put $\tilde{G}^{(2)'} := \bigoplus_{k \in \Lambda} \tilde{G}_{k,k}^{(2)'}$. Then, we have $|\tilde{G}^{(2)'} - G^{(2)}| \leq C_8 \epsilon^{1/2}$ and $[\tilde{G}^{(2)'}, \tilde{G}^{(1)}] = 0$. \blacksquare

3 Estimates for L^2 -Instantons

3.1 Preliminary

Let τ be a complex number such that $\text{Im } \tau > 0$. Let T be a complex torus obtained as the quotient of \mathbb{C} by a lattice $\mathbb{Z} + \mathbb{Z}\tau$. Let z be the standard coordinate of \mathbb{C} . It also gives a local coordinate of a small open subset in T , once we fix a lift of the open subset in \mathbb{C} . We shall use the metric $dz d\bar{z}$ for \mathbb{C} and T unless otherwise specified.

For any open subset $W \subset \mathbb{C}_w$, We use the metric $dw d\bar{w}$ on W , and the metric $dz d\bar{z} + dw d\bar{w}$ on $T \times W$ unless otherwise specified. Let ω denote the associated Kähler form. For $w \in W$, we put $T_w := T \times \{w\} \subset T \times W$.

Let E be a complex C^∞ -vector bundle on $T \times W$ with a hermitian metric h and a unitary connection ∇ . Let $F(\nabla)$ denote the curvature of ∇ . We shall often denote it simply by F . The $(0, 1)$ -part and the $(1, 0)$ -part of ∇ are denoted by $\bar{\partial}_E$ and ∂_E , respectively. The restrictions of (E, h) to T_w are denoted by (E_w, h_w) .

Recall that (E, ∇, h) is called an instanton, if $\Lambda_\omega F(\nabla) = 0$. For the expression $F(\nabla) = F_{z\bar{z}} dz d\bar{z} + F_{z\bar{w}} dz d\bar{w} + F_{w\bar{z}} dw d\bar{z} + F_{w\bar{w}} dw d\bar{w}$, the equation is $F_{z\bar{z}} + F_{w\bar{w}} = 0$. We have the following equalities:

$$(\nabla_z \nabla_{\bar{z}} + \nabla_w \nabla_{\bar{w}}) F_{w\bar{w}} = -(\nabla_z \nabla_{\bar{z}} + \nabla_w \nabla_{\bar{w}}) F_{z\bar{z}} = [F_{z\bar{w}}, F_{w\bar{z}}] \quad (5)$$

$$(\nabla_z \nabla_{\bar{z}} + \nabla_w \nabla_{\bar{w}}) F_{z\bar{w}} = 2[F_{w\bar{w}}, F_{z\bar{w}}], \quad (\nabla_z \nabla_{\bar{z}} + \nabla_w \nabla_{\bar{w}}) F_{w\bar{z}} = 2[F_{w\bar{z}}, F_{w\bar{w}}] \quad (6)$$

3.1.1 Hitchin's equivalence

Let us recall the relation between harmonic bundles on an open subset $W \subset \mathbb{C}_w$ and instantons on $T \times W$ due to Hitchin. Let $(E, \bar{\partial}_E, h, \theta)$ be a harmonic bundle on W . Let $\nabla^0 := \bar{\partial}_E + \partial_E$ be the Chern connection. Let θ^\dagger be the adjoint of θ . Let $p : T \times W \rightarrow W$ be the projection. The pull back $p^*(E, \nabla^0, h)$ is denoted by (E_1, ∇^1, h_1) . We set $\nabla := \nabla^1 + f d\bar{z} - f^\dagger dz$. Then, (E_1, ∇, h_1) is an instanton on $T \times W$.

Conversely, let (E_2, ∇^2, h_2) be a T -equivariant instanton on $T \times W$. By considering T -equivariant sections, we obtain a vector bundle E on W such that $p^*E \simeq E_2$. It is naturally equipped with a connection ∇^0 such that $p^*\nabla_v^0 = \nabla_v^2$, where v denotes a natural horizontal lift of a vector field on W . By using the T -equivariance of ∇^2 , we have the expression $\nabla^2 - p^*\nabla^0 = p^*f d\bar{z} - p^*f^\dagger dz$, where f is a section of $\text{End}(E)$. Then, $(E, \bar{\partial}_E, h, f dz)$ is a harmonic bundle. In summary, we have the following.

Proposition 3.1 (Hitchin) *Harmonic bundles on W naturally correspond to T -equivariant instantons on $T \times W$.* ■

3.2 Local estimate

Let U be a closed disc $\{w \mid |w - w_0| \leq 1\}$ of \mathbb{C} . Let (E, ∇, h) be an instanton on $T \times U$.

Assumption 3.2 *We assume that $|F(\nabla)| \leq \epsilon$ for a given positive small number ϵ . We also impose Assumption 2.14.* ■

Note that we also obtain $|D_{\mathbf{x}}^{k_1} D_w^{k_2} F| \leq C_{k_1, k_2} \epsilon$, where C_{k_1, k_2} is a constant depending only on (k_1, k_2) .

3.2.1 Estimates of the \perp -part of the connection form

Let \mathbf{u} be a partially almost holomorphic frame as in Proposition 2.11. We also assume that $\int_T H(h, \mathbf{u}) = I$, as in §2.3.3. Let A be the connection form of ∇ with respect to \mathbf{u} . Let $\mathcal{B}_{z\bar{z}}$ represent $F_{z\bar{z}}$ with respect to \mathbf{u} . We use $\mathcal{B}_{z\bar{w}}$ and $\mathcal{B}_{w\bar{z}}$ in similar meanings.

Let V be a vector bundle with a hermitian metric h_V on U . Let $\pi : T \times U \rightarrow U$ be the projection. Let $p \geq 2$. For a section f of π^*V on $T \times U$, let $\|f\|_p$ denote the function on U given by $\|f\|_p(w) = \left(\int_{T \times \{w\}} |f|_{h_V}^p \right)^{1/p}$.

Lemma 3.3 *We have $\|A_w^\perp\|_p = O(\|F_{w\bar{z}}^\perp\|_p)$ and $\|\partial_{\bar{w}} A_w^\perp\|_p = O(\|\nabla_{\bar{w}} F_{w\bar{z}}^\perp\|_p) + O(\epsilon \|F_{w\bar{z}}^\perp\|_p)$.*

Proof Because $\partial_w A_{\bar{z}} - \partial_{\bar{z}} A_w + [A_w, A_{\bar{z}}] = \mathcal{B}_{w\bar{z}}$, we have the following equalities:

$$\partial_{\bar{z}} A_w^\perp + [A_{\bar{z}}, A_w^\perp] = -\mathcal{B}_{w\bar{z}}^\perp \quad (7)$$

Then, we obtain the first estimate. We also obtain the following equation:

$$\partial_{\bar{z}} \partial_{\bar{w}} A_w^\perp + [A_{\bar{z}}, \partial_{\bar{w}} A_w^\perp] = -\partial_{\bar{w}} \mathcal{B}_{w\bar{z}}^\perp - [\partial_{\bar{w}} A_{\bar{z}}, A_w^\perp]$$

Because $\partial_{\bar{w}} A_{\bar{z}} = O(\epsilon)$, we obtain the second estimate. ■

Lemma 3.4 *We have $\|A_{\bar{w}}^\perp\|_p = O(\|F_{w\bar{z}}^\perp\|_p + \|F_{z\bar{z}}^\perp\|_p + \|\nabla_{\bar{w}} F_{z\bar{z}}^\perp\|_p)$. We also have*

$$\|\partial_w A_{\bar{w}}^\perp\|_p = O(\|\nabla_{\bar{w}} F_{w\bar{z}}^\perp\|_p + \|F_{w\bar{z}}^\perp\|_p + \|F_{z\bar{z}}^\perp\|_p + \|\nabla_{\bar{w}} F_{z\bar{z}}^\perp\|_p).$$

Proof We set $\tilde{g} := \overline{H(h, \mathbf{u})}$. We have $A_{\bar{w}} = -\tilde{g}^{-1} \overline{({}^t A_w)} \tilde{g} + \tilde{g}^{-1} \partial_{\bar{w}} \tilde{g}$. Hence, the first claim follows from Lemma 3.3, Lemma 2.15 and Lemma 2.18. We have $\partial_w A_{\bar{w}} - \partial_{\bar{w}} A_w + [A_w, A_{\bar{w}}] = \mathcal{B}_{w\bar{w}}$. Hence, we have

$$\|\partial_w A_{\bar{w}}^\perp\|_p = O(\|\partial_{\bar{w}} A_w^\perp\|_p) + O(\|A_{\bar{w}}^\perp\|_p + \|A_w^\perp\|_p) + \|F_{w\bar{w}}^\perp\|_p$$

Then, the second claim follows. ■

3.2.2 Estimate of the \perp -part of the curvature

Let V be a vector bundle with a hermitian metric h_V on $T \times U$. Let $\pi : T \times U \rightarrow U$ be the projection. For a section f of V on $T \times U$, let $\|f\|$ denote the function on U given by $\left(\int_T |f|_{h_V}^2\right)^{1/2}$. For sections f and g of V , let $((f, g))$ denote the function on U given by $\int_T h_V(f, g)$.

Proposition 3.5 *We have the following:*

$$\begin{aligned} \Delta_w \|F_{z\bar{z}}^\perp\|^2 &\leq -\|\nabla_{\bar{z}} F_{z\bar{z}}^\perp\|^2 - \|\nabla_z F_{z\bar{z}}^\perp\|^2 - \|\nabla_{\bar{w}} F_{z\bar{z}}^\perp\|^2 - \|\nabla_w F_{z\bar{z}}^\perp\|^2 \\ &\quad + O\left(\epsilon \|F_{z\bar{z}}^\perp\|^2 + \epsilon \|F_{w\bar{z}}^\perp\| \|F_{z\bar{z}}^\perp\| + \epsilon \|\nabla_{\bar{w}} F_{w\bar{z}}^\perp\| \|F_{z\bar{z}}^\perp\| + \epsilon \|\nabla_z F_{z\bar{z}}^\perp\| \|F_{z\bar{z}}^\perp\|\right) \\ &\quad + O\left(\epsilon \|\nabla_{\bar{w}} F_{z\bar{z}}^\perp\| \|F_{z\bar{z}}^\perp\| + \epsilon \|F_{w\bar{z}}^\perp\|^2 + \epsilon \|F_{w\bar{z}}^\perp\| \|\nabla_{\bar{w}} F_{z\bar{z}}^\perp\|\right) \end{aligned} \quad (8)$$

Proof We have the following equation:

$$\Delta_w |F_{z\bar{z}}^\perp|^2 = -(\nabla_w \nabla_{\bar{w}} F_{z\bar{z}}^\perp, F_{z\bar{z}}^\perp) - (F_{z\bar{z}}^\perp, \nabla_{\bar{w}} \nabla_w F_{z\bar{z}}^\perp) - (\nabla_w F_{z\bar{z}}^\perp, \nabla_w F_{z\bar{z}}^\perp) - (\nabla_{\bar{w}} F_{z\bar{z}}^\perp, \nabla_{\bar{w}} F_{z\bar{z}}^\perp)$$

We have

$$-(\nabla_w \nabla_{\bar{w}} F_{z\bar{z}}^\perp, F_{z\bar{z}}^\perp) = -(\nabla_w \nabla_{\bar{w}} F_{z\bar{z}}, F_{z\bar{z}}^\perp) + (\nabla_w \nabla_{\bar{w}} F_{z\bar{z}}^\circ, F_{z\bar{z}}^\perp)$$

Let us consider the estimate of $(\nabla_w \nabla_{\bar{w}} F_{z\bar{z}}^\circ, F_{z\bar{z}}^\perp)$. The endomorphism $\nabla_w \nabla_{\bar{w}} F_{z\bar{z}}^\circ$ is represented by the following with respect to \mathbf{u} :

$$\partial_w \partial_{\bar{w}} \mathcal{B}_{z\bar{z}}^\circ + [A_w, \partial_{\bar{w}} \mathcal{B}_{z\bar{z}}^\circ] + \partial_w [A_{\bar{w}}, \mathcal{B}_{z\bar{z}}^\circ] + [A_w, [A_{\bar{w}}, \mathcal{B}_{z\bar{z}}^\circ]]$$

We have the following estimates:

$$((\partial_w \partial_{\bar{w}} \mathcal{B}_{z\bar{z}}^\circ, \mathcal{B}_{z\bar{z}}^\perp))_h = O\left(\|\partial_w \partial_{\bar{w}} \mathcal{B}_{z\bar{z}}^\circ\|_h \|\mathcal{B}_{z\bar{z}}^\perp\|_h \|F_{z\bar{z}}^\perp\|_h\right) \quad (9)$$

$$([A_w, \partial_{\bar{w}} \mathcal{B}_{z\bar{z}}^\circ], \mathcal{B}_{z\bar{z}}^\perp)_h = O\left(\|A_w^\perp\|_h \|\partial_{\bar{w}} \mathcal{B}_{z\bar{z}}^\circ\|_h \|\mathcal{B}_{z\bar{z}}^\perp\|_h\right) + O\left(\|[A_w^\circ, \partial_{\bar{w}} \mathcal{B}_{z\bar{z}}^\circ]\|_h \|\mathcal{B}_{z\bar{z}}^\perp\|_h \|F_{z\bar{z}}^\perp\|_h\right) \quad (10)$$

$$\begin{aligned} ([\partial_w A_{\bar{w}}, \mathcal{B}_{z\bar{z}}^\circ], \mathcal{B}_{z\bar{z}}^\perp)_h &= ([\partial_w A_{\bar{w}}^\perp, \mathcal{B}_{z\bar{z}}^\circ], \mathcal{B}_{z\bar{z}}^\perp)_h + ([\partial_w A_{\bar{w}}^\circ, \mathcal{B}_{z\bar{z}}^\circ], \mathcal{B}_{z\bar{z}}^\perp)_h \\ &= O(\|\mathcal{B}_{z\bar{z}}^\circ\|_h \|\partial_w A_{\bar{w}}^\perp\|_h \|\mathcal{B}_{z\bar{z}}^\perp\|_h) + O(\|[\partial_w A_{\bar{w}}^\circ, \mathcal{B}_{z\bar{z}}^\circ]\|_h \|\mathcal{B}_{z\bar{z}}^\perp\|_h \|F_{z\bar{z}}^\perp\|_h) \end{aligned} \quad (11)$$

$$([A_{\bar{w}}, \partial_w \mathcal{B}_{z\bar{z}}^\circ], \mathcal{B}_{z\bar{z}}^\perp)_h = O(\|\partial_w \mathcal{B}_{z\bar{z}}^\circ\|_h \|A_{\bar{w}}^\perp\|_h \|\mathcal{B}_{z\bar{z}}^\perp\|_h) + O(\|[A_{\bar{w}}^\circ, \partial_w \mathcal{B}_{z\bar{z}}^\circ]\|_h \|\mathcal{B}_{z\bar{z}}^\perp\|_h \|F_{z\bar{z}}^\perp\|_h) \quad (12)$$

$$\begin{aligned} ([A_w, [A_{\bar{w}}, \mathcal{B}_{z\bar{z}}^\circ]], \mathcal{B}_{z\bar{z}}^\perp)_h &= O(\|A_w^\perp\|_h \|A_{\bar{w}}^\perp\|_h \|\mathcal{B}_{z\bar{z}}^\circ\|_h \|\mathcal{B}_{z\bar{z}}^\perp\|_h) + O(\|A_w^\perp\|_h \|A_{\bar{w}}^\circ\|_h \|\mathcal{B}_{z\bar{z}}^\circ\|_h \|\mathcal{B}_{z\bar{z}}^\perp\|_h) \\ &\quad + O(\|A_w^\perp\|_h \|A_w^\circ\|_h \|\mathcal{B}_{z\bar{z}}^\circ\|_h \|\mathcal{B}_{z\bar{z}}^\perp\|_h) + O(\|A_w^\circ\|_h \|A_{\bar{w}}^\circ\|_h \|\mathcal{B}_{z\bar{z}}^\circ\|_h \|\mathcal{B}_{z\bar{z}}^\perp\|_h \|F_{z\bar{z}}^\perp\|_h) \end{aligned} \quad (13)$$

We obtain the following estimate for $(\nabla_w \nabla_{\bar{w}} F_{z\bar{z}}^\circ, F_{z\bar{z}}^\perp)$ from (9)–(13) with Lemma 3.3:

$$(\nabla_w \nabla_{\bar{w}} F_{z\bar{z}}^\circ, F_{z\bar{z}}^\perp) = O\left(\epsilon \|F_{z\bar{z}}^\perp\|^2 + \epsilon \|F_{w\bar{z}}^\perp\| \|F_{z\bar{z}}^\perp\| + \epsilon \|\nabla_{\bar{w}} F_{w\bar{z}}^\perp\| \|F_{z\bar{z}}^\perp\| + \epsilon \|\nabla_w F_{z\bar{z}}^\perp\| \|F_{z\bar{z}}^\perp\|\right) \quad (14)$$

We have

$$-(\nabla_w \nabla_{\bar{w}} F_{z\bar{z}}, F_{z\bar{z}}^\perp) = ((\nabla_z \nabla_{\bar{z}} F_{z\bar{z}}, F_{z\bar{z}}^\perp)) + ([F_{z\bar{w}}, F_{w\bar{z}}], F_{z\bar{z}}^\perp) = -(\nabla_{\bar{z}} F_{z\bar{z}}, \nabla_{\bar{z}} F_{z\bar{z}}^\perp) + ([F_{z\bar{w}}, F_{w\bar{z}}], F_{z\bar{z}}^\perp) \quad (15)$$

We have

$$\begin{aligned} -(\nabla_{\bar{z}} F_{z\bar{z}}, \nabla_{\bar{z}} F_{z\bar{z}}^\perp) &= -(\nabla_{\bar{z}} F_{z\bar{z}}^\perp, \nabla_{\bar{z}} F_{z\bar{z}}^\perp) - (\nabla_{\bar{z}} F_{z\bar{z}}^\circ, \nabla_{\bar{z}} F_{z\bar{z}}^\perp) \\ &= -(\nabla_{\bar{z}} F_{z\bar{z}}^\perp, \nabla_{\bar{z}} F_{z\bar{z}}^\perp) + O(\|\nabla_{\bar{z}} F_{z\bar{z}}^\circ\| \|\nabla_{\bar{z}} F_{z\bar{z}}^\perp\| \|F_{z\bar{z}}^\perp\|) \end{aligned} \quad (16)$$

We also have the following:

$$\begin{aligned} (([F_{z\bar{w}}, F_{w\bar{z}}], F_{z\bar{z}}^\perp)) &= O(\|F_{z\bar{w}}^\circ, F_{w\bar{z}}^\circ\| \|F_{z\bar{z}}^\perp\| \|F_{z\bar{z}}^\perp\|) + O(\|F_{z\bar{w}}^\circ\| \|F_{w\bar{z}}^\perp\| \|F_{z\bar{z}}^\perp\|) \\ &\quad + O(\|F_{w\bar{z}}^\circ\| \|F_{z\bar{w}}^\perp\| \|F_{z\bar{z}}^\perp\|) + O(\|F_{z\bar{w}}^\perp\| \|F_{w\bar{z}}^\perp\| \|F_{z\bar{z}}^\perp\|) \end{aligned} \quad (17)$$

We have a similar estimate for the contribution of $-((F_{z\bar{z}}^\perp, \nabla_{\bar{w}} \nabla_w F_{z\bar{z}}^\perp))$. In all, we obtain the claim of Proposition 3.5. \blacksquare

Proposition 3.6 *We have the following inequality:*

$$\begin{aligned} \Delta_w \|F_{z\bar{w}}^\perp\|^2 &\leq -\|\nabla_{\bar{z}} F_{z\bar{w}}^\perp\|^2 - \|\nabla_z F_{z\bar{w}}^\perp\|^2 - \|\nabla_w F_{z\bar{w}}^\perp\|^2 - \|\nabla_{\bar{w}} F_{z\bar{w}}^\perp\|^2 \\ &\quad + O\left(\epsilon \|F_{z\bar{w}}^\perp\| \|F_{z\bar{z}}^\perp\| + \epsilon \|\nabla_{\bar{w}} F_{w\bar{z}}^\perp\| \|F_{z\bar{w}}^\perp\| + \epsilon \|F_{w\bar{z}}^\perp\| \|F_{z\bar{w}}^\perp\| + \epsilon \|\nabla_{\bar{w}} F_{z\bar{z}}^\perp\| \|F_{z\bar{w}}^\perp\|\right) \\ &\quad + O\left(\epsilon \|\nabla_{\bar{z}} F_{z\bar{w}}^\perp\| \|F_{z\bar{z}}^\perp\| + \epsilon \|F_{z\bar{w}}^\perp\| \|F_{w\bar{z}}^\perp\|\right) \end{aligned} \quad (18)$$

Proof We have the following:

$$-\partial_w \partial_{\bar{w}} |F_{z\bar{w}}^\perp|^2 = -|\nabla_{\bar{w}} F_{z\bar{w}}^\perp|^2 - |\nabla_w F_{z\bar{w}}^\perp|^2 - (\nabla_w \nabla_{\bar{w}} F_{z\bar{w}}^\perp, F_{z\bar{w}}^\perp) - (F_{z\bar{w}}^\perp, \nabla_{\bar{w}} \nabla_w F_{z\bar{w}}^\perp) \quad (19)$$

We have

$$-(\nabla_w \nabla_{\bar{w}} F_{z\bar{w}}^\perp, F_{z\bar{w}}^\perp) = -(\nabla_w \nabla_{\bar{w}} F_{z\bar{w}}, F_{z\bar{w}}^\perp) + (\nabla_w \nabla_{\bar{w}} F_{z\bar{w}}^\circ, F_{z\bar{w}}^\perp) \quad (20)$$

Let us look at the contribution of $(\nabla_w \nabla_{\bar{w}} F_{z\bar{w}}^\circ, F_{z\bar{w}}^\perp)$. Let $\mathcal{B}_{z\bar{w}}$ express $F_{z\bar{w}}$ with respect to \mathbf{u} as in the proof of Proposition 3.5. Then, $\nabla_w \nabla_{\bar{w}} F_{z\bar{w}}^\circ$ is represented by the following:

$$\partial_w \partial_{\bar{w}} \mathcal{B}_{z\bar{w}}^\circ + [\partial_w A_{\bar{w}}, \mathcal{B}_{z\bar{w}}^\circ] + [A_{\bar{w}}, \partial_w \mathcal{B}_{z\bar{w}}^\circ] + [A_w, \partial_{\bar{w}} \mathcal{B}_{z\bar{w}}^\circ] + [A_w, [A_{\bar{w}}, \mathcal{B}_{z\bar{w}}^\circ]]$$

We have the following estimates:

$$-((\partial_w \partial_{\bar{w}} \mathcal{B}_{z\bar{w}}^\circ, \mathcal{B}_{z\bar{w}}^\perp))_h = O\left(\|\partial_w \partial_{\bar{w}} \mathcal{B}_{z\bar{w}}^\circ\|_h \|\mathcal{B}_{z\bar{w}}^\perp\|_h \|F_{z\bar{z}}^\perp\|\right) \quad (21)$$

$$(([\partial_w A_{\bar{w}}, \mathcal{B}_{z\bar{w}}^\circ], \mathcal{B}_{z\bar{w}}^\perp))_h = O\left(\|[\partial_w A_{\bar{w}}, \mathcal{B}_{z\bar{w}}^\circ]\|_h \|\mathcal{B}_{z\bar{w}}^\perp\|_h \|F_{z\bar{z}}^\perp\|\right) + O\left(\|\partial_w A_{\bar{w}}^\perp\|_h \|\mathcal{B}_{z\bar{w}}^\circ\|_h \|\mathcal{B}_{z\bar{w}}^\perp\|_h\right) \quad (22)$$

$$(([\partial_w A_{\bar{w}}, \mathcal{B}_{z\bar{w}}^\circ], \mathcal{B}_{z\bar{w}}^\perp))_h = O\left(\|A_{\bar{w}}^\circ\|_h \|\partial_w \mathcal{B}_{z\bar{w}}^\circ\|_h \|\mathcal{B}_{z\bar{w}}^\perp\|_h \|F_{z\bar{z}}^\perp\|\right) + O\left(\|A_{\bar{w}}^\perp\|_h \|\partial_w \mathcal{B}_{z\bar{w}}^\circ\|_h \|\mathcal{B}_{z\bar{w}}^\perp\|_h\right) \quad (23)$$

$$(([\partial_w A_{\bar{w}}, \mathcal{B}_{z\bar{w}}^\circ], \mathcal{B}_{z\bar{w}}^\perp))_h = O\left(\|A_{\bar{w}}^\circ\|_h \|\partial_{\bar{w}} \mathcal{B}_{z\bar{w}}^\circ\|_h \|\mathcal{B}_{z\bar{w}}^\perp\|_h \|F_{z\bar{z}}^\perp\|\right) + O\left(\|A_{\bar{w}}^\perp\|_h \|\partial_{\bar{w}} \mathcal{B}_{z\bar{w}}^\circ\|_h \|\mathcal{B}_{z\bar{w}}^\perp\|_h\right) \quad (24)$$

$$\begin{aligned} (([A_w, [A_{\bar{w}}, \mathcal{B}_{z\bar{w}}^\circ]], \mathcal{B}_{z\bar{w}}^\perp))_h &= O\left(\|A_w^\perp\|_h \|A_{\bar{w}}^\perp\|_h \|\mathcal{B}_{z\bar{w}}^\circ\|_h \|\mathcal{B}_{z\bar{w}}^\perp\|_h\right) + O\left(\|A_w^\circ\|_h \|A_{\bar{w}}^\perp\|_h \|\mathcal{B}_{z\bar{w}}^\circ\|_h \|\mathcal{B}_{z\bar{w}}^\perp\|_h\right) \\ &\quad + O\left(\|A_w^\perp\|_h \|A_{\bar{w}}^\circ\|_h \|\mathcal{B}_{z\bar{w}}^\circ\|_h \|\mathcal{B}_{z\bar{w}}^\perp\|_h\right) + O\left(\|A_w^\circ\|_h \|A_{\bar{w}}^\circ\|_h \|\mathcal{B}_{z\bar{w}}^\circ\|_h \|\mathcal{B}_{z\bar{w}}^\perp\|_h \|F_{z\bar{z}}^\perp\|\right) \end{aligned} \quad (25)$$

Hence, we obtain the following:

$$((\nabla_w \nabla_{\bar{w}} F_{z\bar{w}}^\circ, F_{z\bar{w}}^\perp)) = O\left(\epsilon \|F_{z\bar{w}}^\perp\| \|F_{z\bar{z}}^\perp\|_h + \epsilon \|\nabla_{\bar{w}} F_{w\bar{z}}^\perp\| \|F_{z\bar{w}}^\perp\|_h + \epsilon \|F_{w\bar{z}}^\perp\| \|F_{z\bar{w}}^\perp\|_h + \epsilon \|\nabla_{\bar{w}} F_{z\bar{z}}^\perp\|_h \|F_{z\bar{w}}^\perp\|_h\right) \quad (26)$$

We have

$$\begin{aligned} -((\nabla_w \nabla_{\bar{w}} F_{z\bar{w}}, F_{z\bar{w}}^\perp)) &= ((\nabla_z \nabla_{\bar{z}} F_{z\bar{w}}, F_{z\bar{w}}^\perp)) - 2(([F_{w\bar{w}}, F_{z\bar{w}}], F_{z\bar{w}}^\perp)) \\ &= -((\nabla_{\bar{z}} F_{z\bar{w}}, \nabla_{\bar{z}} F_{z\bar{w}}^\perp)) - 2(([F_{w,\bar{w}}, F_{z\bar{w}}], F_{z\bar{w}}^\perp)) \end{aligned} \quad (27)$$

We have

$$\begin{aligned} -(\nabla_{\bar{z}} F_{z\bar{w}}, \nabla_{\bar{z}} F_{z\bar{w}}^\perp) &= -(\nabla_{\bar{z}} F_{z\bar{w}}, \nabla_{\bar{z}} F_{z\bar{w}}^\perp) - (\nabla_{\bar{z}} F_{z\bar{w}}^\circ, \nabla_{\bar{z}} F_{z\bar{w}}^\perp) \\ &= -(\nabla_{\bar{z}} F_{z\bar{w}}^\perp, \nabla_{\bar{z}} F_{z\bar{w}}^\perp) + O\left(\|\nabla_{\bar{z}} F_{z\bar{w}}^\circ\| \|\nabla_{\bar{z}} F_{z\bar{w}}^\perp\| \|F_{z\bar{z}}^\perp\|\right) \end{aligned} \quad (28)$$

We also have

$$\begin{aligned} ([F_{w\bar{w}}, F_{z\bar{w}}], F_{z\bar{w}}^\perp) &= O\left(\|F_{w\bar{w}}^\perp\| \|F_{z\bar{w}}^\perp\| \|F_{z\bar{w}}^\perp\|\right) + O\left(\|F_{w\bar{w}}^\circ\| \|F_{z\bar{w}}^\perp\| \|F_{z\bar{w}}^\perp\|\right) \\ &\quad + O\left(\|F_{w\bar{w}}^\perp\| \|F_{z\bar{w}}^\circ\| \|F_{z\bar{w}}^\perp\|\right) + O\left(\|F_{w\bar{w}}^\circ, F_{z\bar{w}}^\circ\| \|F_{z\bar{w}}^\perp\| \|F_{z\bar{z}}^\perp\|\right) \end{aligned} \quad (29)$$

We have a similar estimate for the contribution of $-(F_{z\bar{w}}^\perp, \nabla_{\bar{w}} \nabla_w F_{z\bar{w}}^\perp)$. In all, we obtain the desired estimate (18). \blacksquare

Proposition 3.7 *There exist $C > 0$ and $\epsilon_0 > 0$ such that the following inequality holds if $\epsilon < \epsilon_0$:*

$$\begin{aligned} \Delta_w (\|F_{z\bar{z}}^\perp\|^2 + \|F_{z\bar{w}}^\perp\|^2) &\leq -C\left(\|F_{z\bar{z}}^\perp\|^2 + \|F_{z\bar{w}}^\perp\|^2\right) - C\left(\|\nabla_z F_{z\bar{z}}^\perp\|^2 + \|\nabla_{\bar{z}} F_{z\bar{z}}^\perp\|^2 + \|\nabla_w F_{z\bar{z}}^\perp\|^2 + \|\nabla_{\bar{w}} F_{z\bar{z}}^\perp\|^2\right) \\ &\quad - C\left(\|\nabla_z F_{z\bar{w}}^\perp\|^2 + \|\nabla_{\bar{z}} F_{z\bar{w}}^\perp\|^2 + \|\nabla_w F_{z\bar{w}}^\perp\|^2 + \|\nabla_{\bar{w}} F_{z\bar{w}}^\perp\|^2\right) \end{aligned} \quad (30)$$

Proof There exist $C_1 > 0$ such that $\|\nabla_z s\| \geq C_1 \|s\|$ and $\|\nabla_{\bar{z}} s\| \geq C_1 \|s\|$ for any section of $\text{End}(E)$ such that $s = s^\perp$. Then, the claim follows from Proposition 3.5 and Proposition 3.6. \blacksquare

3.2.3 Higher derivative

Assume that $\|F_{z\bar{z}}^\perp\|^2 + \|F_{z\bar{w}}^\perp\|^2 \leq \delta^2$ for some $\delta \ll \epsilon$. For $\rho < 1$, we set $U(\rho) = \{w \mid |w - w_0| \leq \rho\} \subset U$.

Proposition 3.8 *For any k, p , there exists $C > 0$ such that*

$$\|F_{z\bar{z}}^\perp\|_{L_k^p(T \times U(\rho))} \leq C\delta, \quad \|F_{z\bar{w}}^\perp\|_{L_k^p(T \times U(\rho))} \leq C\delta.$$

Proof It can be shown by a standard bootstrapping argument. We give only an indication. We take $\rho < \rho' < 1$. In the following, we shall replace ρ' with smaller one. Let κ denote z, \bar{z}, w and \bar{w} . By Proposition 3.7, we obtain $\|\nabla_\kappa F_{z\bar{z}}^\perp\|_{L^2(T \times U(\rho'))} = O(\delta)$ and $\|\nabla_\kappa F_{z\bar{w}}^\perp\|_{L^2(T \times U(\rho'))} = O(\delta)$.

With respect to the frame \mathbf{u} , the endomorphism $-\nabla_w \nabla_{\bar{w}} F_{z\bar{z}}$ is represented by

$$-\partial_w \partial_{\bar{w}} \mathcal{B}_{z\bar{z}} - [\partial_w A_{\bar{w}}, \mathcal{B}_{z\bar{z}}] - [A_{\bar{w}}, \partial_w \mathcal{B}_{z\bar{z}}] + [A_w, \partial_{\bar{w}} \mathcal{B}_{z\bar{z}}] + [A_w, [A_{\bar{w}}, \mathcal{B}_{z\bar{z}}]], \quad (31)$$

and the endomorphism $-\nabla_z \nabla_{\bar{z}} F_{z\bar{z}}$ is represented by

$$-\partial_z \partial_{\bar{z}} \mathcal{B}_{z\bar{z}} - [\partial_z A_{\bar{z}}, \mathcal{B}_{z\bar{z}}] - [A_{\bar{z}}, \partial_z \mathcal{B}_{z\bar{z}}] + [A_z, \partial_{\bar{z}} \mathcal{B}_{z\bar{z}}] + [A_z, [A_{\bar{z}}, \mathcal{B}_{z\bar{z}}]]. \quad (32)$$

The sum of (31) and (32) is equal to $[\mathcal{B}_{z\bar{w}}, \mathcal{B}_{w\bar{z}}]$. By looking at the \perp -part of the equation, we obtain the following equation:

$$\text{The } \perp\text{-part of (31) + The } \perp\text{-part of (32)} = [\mathcal{B}_{z\bar{w}}, \mathcal{B}_{w\bar{z}}]^\perp \quad (33)$$

By using Lemma 3.3 and Lemma 3.4, we obtain $\|(\partial_w \partial_{\bar{w}} + \partial_z \partial_{\bar{z}}) \mathcal{B}_{z\bar{z}}^\perp\|_{L^2(T \times U(\rho'))} = O(\delta)$. Similarly, we obtain $\|(\partial_w \partial_{\bar{w}} + \partial_z \partial_{\bar{z}}) \mathcal{B}_{z\bar{w}}^\perp\|_{L^2(T \times U(\rho'))} = O(\delta)$. It follows that

$$\|F_{z\bar{z}}^\perp\|_{L^4(T \times U(\rho'))} + \|F_{z\bar{w}}^\perp\|_{L^4(T \times U(\rho'))} = O(\delta)$$

$$\|\nabla_\kappa F_{z\bar{z}}^\perp\|_{L^4(T \times U(\rho'))} + \|\nabla_\kappa F_{z\bar{w}}^\perp\|_{L^4(T \times U(\rho'))} = O(\delta)$$

By using Lemma 3.3, Lemma 3.4 and (33), we obtain $\|(\partial_w \partial_{\bar{w}} + \partial_z \partial_{\bar{z}}) \mathcal{B}_{z\bar{z}}^\perp\|_{L^4(T \times U(\rho'))} = O(\delta)$. Similarly, we obtain $\|(\partial_w \partial_{\bar{w}} + \partial_z \partial_{\bar{z}}) \mathcal{B}_{z\bar{w}}^\perp\|_{L^4(T \times U(\rho'))} = O(\delta)$. By the same argument, we obtain the following for any p :

$$\|F_{z\bar{z}}^\perp\|_{L^p(T \times U(\rho'))} + \|F_{z\bar{w}}^\perp\|_{L^p(T \times U(\rho'))} + \|\nabla_\kappa F_{z\bar{z}}^\perp\|_{L^p(T \times U(\rho'))} + \|\nabla_\kappa F_{z\bar{w}}^\perp\|_{L^p(T \times U(\rho'))} = O(\delta)$$

$$\|(\partial_w \partial_{\bar{w}} + \partial_z \partial_{\bar{z}}) \mathcal{B}_{z\bar{z}}^\perp\|_{L^p(T \times U(\rho'))} + \|(\partial_w \partial_{\bar{w}} + \partial_z \partial_{\bar{z}}) \mathcal{B}_{z\bar{w}}^\perp\|_{L^p(T \times U(\rho'))} = O(\delta)$$

Namely, we obtain $\|F_{z\bar{z}}^\perp\|_{L_2^p(T \times U(\rho'))} + \|F_{z\bar{w}}^\perp\|_{L_2^p(T \times U(\rho'))} = O(\delta)$.

By the argument in Lemma 3.3, we obtain $\|A_w^\perp\|_{L_2^p} = O(\delta)$. By the argument in Lemma 3.4, we obtain $\|A_{\bar{w}}^\perp\|_{L_1^p} = O(\delta)$. By the relation $\partial_w A_{\bar{w}} - \partial_{\bar{w}} A_w + [A_w, A_{\bar{w}}] = \mathcal{B}_{w\bar{w}}$, we obtain $\|\partial_w A_{\bar{w}}^\perp\|_{L_1^p} = O(\delta)$. We also have $\|A_z^\perp\|_{L_2^p} = O(\delta)$, which follows from Lemma 2.18. Then, we obtain

$$\|(\partial_w \partial_{\bar{w}} + \partial_z \partial_{\bar{z}}) \mathcal{B}_{z\bar{z}}^\perp\|_{L_1^p(T \times U(\rho'))} = O(\delta)$$

Hence, we obtain $\|\mathcal{B}_{z\bar{z}}^\perp\|_{L_3^p(T \times U(\rho'))} = O(\delta)$. Similarly, we obtain $\|\mathcal{B}_{z\bar{w}}^\perp\|_{L_3^p(T \times U(\rho'))} = O(\delta)$. By the inductive argument, we obtain $\|\mathcal{B}_{z\bar{z}}^\perp\|_{L_k^p(T \times U(\rho'))} + \|\mathcal{B}_{z\bar{w}}^\perp\|_{L_k^p(T \times U(\rho'))} = O(\delta)$ for any k . \blacksquare

Corollary 3.9 *For any k and p , there exists $C > 0$ such that $\|H(h, \mathbf{u})^\perp\|_{L_k^p(T \times U(\rho))} \leq C\delta$.*

Proof It follows from Proposition 3.8 and Lemma 2.18. \blacksquare

3.3 Global estimate

3.3.1 Preliminary

For $R > 0$, we set $Y_R := \{w \in \mathbb{C} \mid |w| \geq R\}$ and $X_R := T \times Y_R$. An instanton (E, ∇, h) is called L^2 , if the curvature $F := F(\nabla)$ is L^2 . We study the behaviour of L^2 -instantons around infinity. We suppose that (E, ∇, h) is an L^2 -instanton in this subsection.

Lemma 3.10 *There exists a constant $C_1 > 0$, which is independent of (E, ∇, h) such that*

$$|F(z, w)| \leq C_1(|w| - R)^{-1} \left(\int_{X_R} |F|^2 \right)^{1/2}.$$

In particular, if $|w| > 2R$, we have $|F(z, w)| \leq 2C_1|w|^{-1} \left(\int_{X_R} |F|^2 \right)^{1/2}$.

Proof It follows from the estimate due to Uhlenbeck [51]. \blacksquare

Corollary 3.11 *There exists $R_0 > 0$ such that $(E_w, \bar{\partial}_{E_w})$ is semistable of degree 0 for any w with $|w| \geq R_0$.*

Proof It follows from Lemma 2.7 and Lemma 3.10. \blacksquare

Because we are interested in the behaviour around infinity, we may assume that $(E_w, \bar{\partial}_{E_w})$ are semistable of degree 0 for any $w \in Y_R$, from the beginning.

Lemma 3.12 *For any $\delta > 0$, there exists $R_1 > 0$ such that $|F(z, w)| \leq \delta|w|^{-1}$ if $|w| \geq R_1$.*

Proof If R_2 is sufficiently large, we have $\left(\int_{X_{R_2}} |F|^2 \right)^{1/2} < \delta(2C_1)^{-1}$. Then, if $|w| > R_2$, we have $|F(z, w)| \leq (|w| - R_2)^{-1}\delta/2$. We can find R_1 such that $(|w| - R)^{-1}/2 \leq |w|^{-1}$ if $|w| > R_1$. \blacksquare

3.3.2 Prolongation of the spectral curve

We consider the relative Fourier-Mukai transform $\text{RFM}_-(E, \bar{\partial}_E)$, which is a coherent sheaf on $T^\vee \times Y_R$. The support is relatively 0-dimensional over Y_R , denoted by $\mathcal{S}p(E)$. It is called the spectral curve of $(E, \bar{\partial}_E)$. Let \bar{Y}_R be the closure of Y_R in \mathbb{P}^1 , i.e., $\bar{Y}_R = Y_R \cup \{\infty\}$.

Theorem 3.13 *$\mathcal{S}p(E)$ is extended to a closed subvariety $\overline{\mathcal{S}p}(E)$ in $T^\vee \times \bar{Y}_R$.*

Proof Let ρ denote the rank of E . We have the holomorphic map $\varphi : Y_R \rightarrow \text{Sym}^\rho T^\vee$ induced by $\mathcal{S}p(E)$. We have only to show that it is extended to a holomorphic map $\overline{Y}_R \rightarrow \text{Sym}^\rho T^\vee$. We fix a closed immersion $\text{Sym}^\rho T^\vee \subset \mathbb{P}^N$ for a sufficiently large N , and we regard φ as a holomorphic map $Y_R \rightarrow \mathbb{P}^N$. Let d denote the distance of \mathbb{P}^N , induced by the Fubini-Study metric.

We use a polar coordinate $w = r e^{\sqrt{-1}\theta}$ for Y_R . We use the isomorphism $\mathbb{R}^2/\mathbb{Z} \simeq T$ given by $(x, y) \mapsto x + \tau y$. We take $y_0 \in \mathbb{R}/\mathbb{Z}$ and $r_0 \geq R$. We set $H(y_0, r_0) := \{(x, y_0, r_0, \theta) \in X_R\}$. Let \mathbf{v} be an orthonormal frame of E on $H(y_0, r_0)$ such that $\nabla(\partial_x)\mathbf{v} = 0$ and $\nabla(\partial_\theta)\mathbf{v}_{(0, \theta)} = 0$. Let $G(\theta) : S^1 \rightarrow U(\rho)$ be determined by $\mathbf{v}_{|(1, \theta)} = \mathbf{v}_{|(0, \theta)} G(\theta)$. We have $\partial_\theta G(\theta) = \Theta(\partial_x, \partial_\theta)(0, y_0, r_0, \theta)$. Recall that, for any $\delta > 0$, there exists $R_0 > 0$ such that $|F| < \delta |w|^{-1}$ on X_{R_0} . Hence, if $r_0 > R_0$, we have $|\partial_\theta G(\theta)| \leq \delta$. We obtain that $|G(\theta_1) - G(\theta_2)| \leq C_1 \delta$.

By using the above estimate with Corollary 2.13, we obtain that $d(\varphi(r_0, \theta_1), \varphi(r_0, \theta_2)) \leq C_2 \delta$. Hence, by Theorem 3.21 below, we obtain that φ is extended on \overline{Y}_R . \blacksquare

The intersection $\overline{\mathcal{S}p}(E) \cap (T^\vee \times \{\infty\})$ is denoted by $\mathcal{S}p_\infty(E)$.

3.3.3 Asymptotic decay

We fix a lift of $\overline{\mathcal{S}p}(E)$ to a closed subvariety $\overline{\mathcal{S}p}(E)_1 \subset \overline{Y}_R \times \mathbb{C}_\zeta$, which induces an action of ζ on $\text{RFM}_-(E, \overline{\partial}_E)$. (See §2.1.) Let f_ζ be the corresponding holomorphic endomorphism of E . We set $\overline{\partial}_0 := \overline{\partial}_E - f_\zeta d\bar{z}$, which gives a holomorphic structure of E . For each w , the restriction of $\mathcal{E}' = (E, \overline{\partial}_0)$ to $T_z \times \{w\}$ is holomorphically trivial. It is naturally isomorphic to $p^*p_*(\mathcal{E}')$, where $p : X_R \rightarrow Y_R$ denotes the natural projection. We obtain the decomposition $h = h^\circ + h^\perp$ as in §2.3.

Theorem 3.14 *For any polynomial $P(t_1, t_2, t_3, t_4)$, there exists $C > 0$ such that*

$$P(\nabla_z, \nabla_{\bar{z}}, \nabla_w, \nabla_{\bar{w}})h^\perp = O(\exp(-C|w|)).$$

Proof By Lemma 3.10 and Theorem 3.13, there exists $R_1 > 0$, such that Assumption 3.2 is satisfied for $(E, \nabla, h)|_{X_{R_1}}$. In particular, we can apply Proposition 3.7 to $(E, \nabla, h)|_{X_{R_1}}$. We obtain the following inequality for some $C_1 > 0$:

$$\Delta_w(\|F_{z\bar{z}}^\perp\|^2 + \|F_{z\bar{w}}^\perp\|^2) \leq -C_1(\|F_{z\bar{z}}^\perp\|^2 + \|F_{z\bar{w}}^\perp\|^2)$$

We obtain the following lemma by a standard argument.

Lemma 3.15 *We have $\|F_{z\bar{z}}^\perp\|^2 + \|F_{z\bar{w}}^\perp\|^2 = O(\exp(-C_2|w|))$ for some $C_2 > 0$.*

Proof This is a variant of a lemma of Ahlfors ([2], [46]). We give only an indication. We put $G := \|F_{z\bar{z}}^\perp\|^2 + \|F_{z\bar{w}}^\perp\|^2$. We put $f_\epsilon := C_3 \exp(-2C_1^{1/2}|w|) + \epsilon$, where $\epsilon > 0$ and $C_3 > 0$. We have the inequality $\Delta_w f_\epsilon \geq -C_1 f_\epsilon$. If C_3 is sufficiently large, we have $f_\epsilon > G$ on $\{|w| = R_1\}$. For each $\epsilon > 0$, we have $f_\epsilon > G$ outside a compact subset. We put $U := \{w \mid f_\epsilon(w) < G(w)\}$. Then, U is relatively compact, and we have $f_\epsilon = G$ on the boundary of U . On U , we have $\Delta_w(G - f_\epsilon) \leq -C(G - f_\epsilon) \leq 0$. By the maximum principle, we have $\sup_U(G - f_\epsilon) \leq \max_{\partial U}(G - f_\epsilon) = 0$. Hence, we obtain that U is empty. It means $G \leq f_\epsilon$ on Y_R for any ϵ . We obtain the desired inequality by taking the limit $\epsilon \rightarrow 0$. \blacksquare

Then, the claim of Theorem 3.14 follows from Corollary 3.9. \blacksquare

3.3.4 Reduction to asymptotic harmonic bundles

Let $p : X_R \rightarrow Y_R$ denote the projection. By using the push-forward of \mathcal{O} -modules, we obtain a holomorphic vector bundle $V := p_*\mathcal{E}'$ on Y_R . It is equipped with a Higgs field $\theta_V := f_\zeta dw$. For any $s_i \in V|_w$ ($i = 1, 2$), the corresponding holomorphic section of $\mathcal{E}'|_{T_w}$ is denoted by \tilde{s}_i . We set $h_V(s_1, s_2) := \int_T h(\tilde{s}_1, \tilde{s}_2)$. We have the Chern connection $\overline{\partial}_V + \partial_V$ with respect to h_V . Let θ_V^\dagger denote the adjoint of θ_V .

Proposition 3.16 *There exists $C > 0$ such that the following holds:*

$$F(h_V) + [\theta_V, \theta_V^\dagger] = O(\exp(-C|w|)). \quad (34)$$

Proof We identify $p^*V = \mathcal{E}'$. According to Theorem 3.14, the difference $h - p^*h_V$ and its derivatives are $O(\exp(-C_1|w|))$. (The constant C_1 may depend on the order of derivatives.) We also have $\overline{\partial}_E = p^*\overline{\partial}_V + f_\zeta d\bar{z}$. Hence, $(p^*V, p^*\overline{\partial}_V + f_\zeta d\bar{z}, p^*h_V)$ satisfies $\Lambda_\omega F(p^*h_V) = O(\exp(-C_2|w|))$, which is equivalent to (34). \blacksquare

3.3.5 Estimate of the curvature

Theorem 3.17 *There exists $\rho > 0$ such that the following holds:*

$$F(h) = O\left(\frac{dz d\bar{z}}{|w|^2(-\log|w|)^2}\right) + O\left(\frac{dw d\bar{w}}{|w|^2(-\log|w|)^2}\right) + O\left(\frac{dw d\bar{z}}{|w|^{1+\rho}}\right) + O\left(\frac{dz d\bar{w}}{|w|^{1+\rho}}\right)$$

Proof We shall use an estimate for asymptotic harmonic bundles explained in §3.6. Let $\varphi : \Delta_u = \{|u| < R^{-1/e}\} \rightarrow \bar{Y}_R$ be given by $\varphi(u) = u^e$. For the expression $\theta = f_\zeta dw = f_\zeta(-eu^{-e-1}du)$, according to Theorem 3.13, the spectral curve $\mathcal{S}p(f_\zeta) \subset \mathbb{C} \times Y_R$ is contained in $\{|\zeta| \leq R'\} \times Y_R$, and the closure in $\mathbb{C} \times \bar{Y}_R$ is a complex variety. Hence, we may assume that $\varphi^*(E, \bar{\partial}_E, \theta)$ has the following decomposition as in (35):

$$\varphi^*E = \bigoplus_{\mathbf{a} \in \text{Irr}(\varphi^*\theta_V)} E_{\mathbf{a}}$$

Moreover, we have $\deg_{u^{-1}} \mathbf{a} \leq e$ for any $\mathbf{a} \in \text{Irr}(\varphi^*\theta_V)$.

We set $(V', \bar{\partial}_{V'}, \theta_{V'}, h') := \varphi^{-1}(V, \bar{\partial}_V, \theta_V, h_V)$. According to Proposition 3.16, it satisfies (36). By Corollary 3.25, we have

$$|F(h_V)|_h = O(|u|^{-2}(\log|u|)^{-2} du d\bar{u})$$

Hence, we have $|F(h)_{w\bar{w}}|_h = |F(h)_{z\bar{z}}|_h = O(|w|^{-2}(\log|w|)^{-2})$.

We take a frame \mathbf{v} of $\mathcal{P}_a V'$ as in §3.6.2. Let Θ be determined by $\varphi^* f_\zeta \mathbf{v} = \mathbf{v} \Theta$. Let C_w be determined by $\varphi^*(\partial_w) \mathbf{v} = \mathbf{v} C_w$. We have $\varphi^*(\partial_w f_\zeta) \mathbf{v} = \mathbf{v} (\varphi^*(\partial_w) \Theta + [C_w, \Theta])$. We have the expression

$$\Theta = \bigoplus ((\varphi^* \partial_w \mathbf{a} - e^{-1} \alpha u^e) I_{\mathbf{a}, \alpha} - e^{-1} u^e \Theta_{\mathbf{a}, \alpha}),$$

where the entries of $\Theta_{\mathbf{a}, \alpha}$ are holomorphic at $u = 0$. The norm of the endomorphism determined by \mathbf{v} and $\Theta_{\mathbf{a}, \alpha}$ is $O((\log|w|)^{-1})$ by Proposition 3.24. Note that $\varphi^*(\partial_w) = -e^{-1} u^{e+1} \partial_u$ and $\varphi^*(\partial_w^2) \mathbf{a} = O(|\varphi^*(w)|^{-1-\rho})$ for some $\rho > 0$. Hence, the contribution of $\varphi^*(\partial_w) \Theta$ to $\varphi^*(\partial_w f)$ is dominated as $O(\varphi^* |w|^{-1-\rho})$ for some $\rho > 0$. Let G_w be the endomorphism determined by \mathbf{v} and C_w . By using Lemma 3.27, we obtain $[G_w, \varphi^* f_\zeta] = O(\varphi^* |w|^{-2})$. Hence, we obtain $|\partial_w f_\zeta|_{h_V} = O(|w|^{-1-\rho})$ for some $\rho > 0$. Then, we obtain $|F_{z\bar{w}}|_h = |F_{w\bar{z}}|_h = O(|w|^{-1-\rho})$ for some $\rho > 0$. \blacksquare

Corollary 3.18 *$(E, \bar{\partial}_E, h)$ is acceptable, i.e., the curvature $F(h)$ is bounded with respect to h and the Poincaré metric $|w|^{-2}(\log|w|)^{-2} dw d\bar{w} + dz d\bar{z}$ on X_R around $T \times \{\infty\}$.* \blacksquare

3.3.6 Prolongation to a filtered bundle

We set $\bar{X}_R := T \times \bar{Y}_R$. Because $(E, \bar{\partial}_E, h)$ is acceptable, we obtain the following from Theorem 21.31 of [38].

Corollary 3.19 *The holomorphic vector bundle $(E, \bar{\partial}_E)$ is naturally extended to a filtered bundle $\mathcal{P}_* E$ on $(\bar{X}_R, T \times \{\infty\})$.* \blacksquare

We obtain the spectral curve $\mathcal{S}p(\mathcal{P}_a E) \subset T^\vee \times \bar{Y}_R$ of $\mathcal{P}_a E$. It is equal to $\bar{\mathcal{S}p}(E)$ in Theorem 3.13, and independent of the choice of $a \in \mathbb{R}$.

3.4 An estimate in a variant case

We continue to use the notation in §3.3. Let (E, ∇, h) be an instanton on X_R . Let $F = F(\nabla)$ be its curvature. We suppose the following:

- For any $\delta > 0$, there exists $R_\delta > 0$ such that $|F(z, w)|_h \leq \delta$ for any $|w| \geq R_\delta$. In particular, we obtain $\mathcal{S}p(E, \bar{\partial}_E) \subset T^\vee \times Y_{R_\delta}$, if δ is sufficiently small.
- The closure of $\mathcal{S}p(E)$ in $T^\vee \times \bar{Y}_{R_\delta}$ is a complex subvariety.

We denote the closure by $\bar{\mathcal{S}p}(E)$, and we set $\mathcal{S}p_\infty(E) := \bar{\mathcal{S}p}(E) \cap (T^\vee \times \{\infty\})$. We obtain the following theorem.

Theorem 3.20 *Under the assumption, (E, ∇, h) is an L^2 -instanton.*

Proof By the assumption, there exists $R_1 > 0$, such that Assumption 3.2 is satisfied for $(E, \nabla, h)|_{X_{R_1}}$. In particular, we can apply Proposition 3.7 to $(E, \nabla, h)|_{X_{R_1}}$. We obtain the estimate as in Theorem 3.14 by the same argument. Then, we obtain estimates as in Proposition 3.16 and Theorem 3.17 by the same arguments. In particular, (E, ∇, h) is an L^2 -instanton. \blacksquare

Theorem 3.20 implies that we can replace the L^2 -condition with a weaker one, under the assumption that the spectral curve is extended in an complex analytic way.

3.5 Extension of holomorphic maps on a punctured disc

In this subsection, we give a proof of a rather general extension property of some holomorphic functions from a punctured disc to a complex projective space, which is used in the proof of Theorem 3.13.

Let d be a distance on \mathbb{P}^n induced by the Fubini-Study metric. For any $P \in \mathbb{P}^n$ and $\epsilon > 0$, let $B(P, \epsilon)$ denote $\{Q \in \mathbb{P}^n \mid d(P, Q) < \epsilon\}$. Let $S_r := \{w \in \mathbb{C} \mid |w| = r\}$ for $0 < r < 1$.

Theorem 3.21 *Let $\varphi : \Delta^* \rightarrow \mathbb{P}^n$ be a holomorphic map with the following property.*

(P) *For any $\epsilon > 0$, there exists $R_0 > 0$ such that, for any $r < R_0$, the condition $P \in \varphi(S_r)$ implies $\varphi(S_r) \subset B(P, \epsilon)$.*

Then, φ is extended to a holomorphic map $\tilde{\varphi} : \Delta \rightarrow \mathbb{P}^n$.

Proof Let us consider the hypersurface $H = \{[0 : \zeta_1 : \dots : \zeta_n]\} \subset \mathbb{P}^n$. We naturally identify $\mathbb{P}^n \setminus H$ with $\mathbb{C}^n = \{(z_1, \dots, z_n)\}$ by $z_i := \zeta_i / \zeta_0$. For any $C > 0$, we put $U_C := \mathbb{P}^n \setminus \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \sum |z_i|^2 \leq C\}$. For any $0 < r_1 < r_2$, we set $S_{r_1, r_2} := \{w \in \mathbb{C} \mid r_1 \leq |w| \leq r_2\}$.

Let $\varphi : \Delta^* \rightarrow \mathbb{P}^n$ be a holomorphic map satisfying the assumption of Theorem 3.21. Fix any $C > 0$. There exists $K > 0$ such that for any $P, Q \in \mathbb{P}^n \setminus U_{1000C}$, we have $K^{-1}d(P, Q) \leq d_{\mathbb{C}^n}(P, Q) \leq Kd(P, Q)$, where $d_{\mathbb{C}^n}$ is the Euclidean distance of \mathbb{C}^n . We take a sufficiently small $\epsilon > 0$ such that $\bigcup_{P \in U_{700C}} B(P, \epsilon)$ is relatively compact in U_{500C} . We also assume that $K\epsilon$ is sufficiently smaller than C . For such ϵ , we take $R_0 > 0$ as in the condition of Theorem 3.21.

Lemma 3.22 *Let $0 < r_1 < r_2 < R_0$. If $\varphi(S_{r_1}) \cup \varphi(S_{r_2}) \subset U_{500C}$, then $\varphi(S_{r_1, r_2}) \subset U_C$.*

Proof We assume that there exists $\varphi(S_s) \not\subset U_C$ for some $s \in [r_1, r_2]$, and we shall derive a contradiction. We set $\mathcal{I} := \{r_1 \leq r \leq s \mid \varphi(S_r) \subset U_{500C}\} \neq \emptyset$. We put $\rho := \sup \mathcal{I}$.

Let us observe that $\varphi(S_r) \cap U_{700C} = \emptyset$ for any $r \in [\rho, s]$. Indeed, assume that $\varphi(S_r) \cap U_{700C} \neq \emptyset$, then there exists $\rho' \in]r, s]$ such that $\varphi(S_{\rho'}) \cap U_{700C} \neq \emptyset$, and hence $\varphi(S_{\rho'}) \subset U_{500C}$. It contradicts with the choice of ρ , and hence we can conclude that $\varphi(S_r) \cap U_{700C} = \emptyset$.

We take $r_3 < \rho$ such that (i) $\varphi(S_{r_3}) \subset U_{500C}$, (ii) we have $\varphi(S_r) \cap U_{1000C} = \emptyset$ for any $r \in [r_3, \rho]$. Note that we also have $\varphi(S_r) \cap U_{1000C} = \emptyset$ for any $r \in [r_3, s]$ by the above remark. Similarly, we can take $r_4 \in]s, r_2]$ such that (i) $\varphi(S_{r_4}) \subset U_{500C}$, (ii) we have $\varphi(S_r) \cap U_{1000C} = \emptyset$ for any $r \in [s, r_4]$.

Let φ_{r_3, r_4} denote the restriction of φ to S_{r_3, r_4} . Because $\text{Im } \varphi_{r_3, r_4} \cap U_{1000C} = \emptyset$, we can regard φ_{r_3, r_4} as a holomorphic map $S_{r_3, r_4} \rightarrow \mathbb{C}^n$.

Let ψ be the composite of φ_{r_3, r_4} and the projection of $\mathbb{C}_{z_1, \dots, z_n}^n$ to \mathbb{C}_{z_1} . By a unitary coordinate change, we may assume that $\psi(S_{r_j})$ ($j = 3, 4$) are contained in $\{|z_1| \geq 200C\}$. For any $a > 0$ and $Q \in \mathbb{C}$, let $B_{\mathbb{C}}(Q, a) := \{Q' \in \mathbb{C} \mid d_{\mathbb{C}}(Q, Q') < a\}$. By applying a rotation, we may assume that $\psi(S_{r_3}) \subset B_{\mathbb{C}}(P_1, 2K\epsilon)$ for a point P_1 with $\arg(P_1) = \pi/4$. We have P_2 such that $\psi(S_{r_4}) \subset B_{\mathbb{C}}(P_2, 2K\epsilon)$. We consider the cases (i) $\pi/4 \leq \arg(P_2) \leq 3\pi/4$, (ii) $3\pi/4 \leq \arg(P_2) \leq 5\pi/4$. The other cases can be argued similarly. We use the coordinate $x + \sqrt{-1}y$ of the target space \mathbb{C} .

(i) Both $\psi(S_{r_3})$ and $\psi(S_{r_4})$ are contained in $\{x + \sqrt{-1}y \mid y \geq 50C\}$. By applying the maximum principle to the harmonic function $-y \circ \psi$, we obtain $\psi(S_{r_3, r_4}) \subset \{x + \sqrt{-1}y \mid y \geq 50C\}$.

(ii) Let $P := 10C(-1 + \sqrt{-1})$. First, we consider the case $P \in \text{Im}(\psi)$. There exists $r \in]r_3, r_4[$ such that $\varphi(S_r) \subset B_{\mathbb{C}}(P, K\epsilon)$. By using the maximum principle of harmonic functions $x \circ \psi$ and $-y \circ \psi$, we

obtain $\psi(S_{r_3,r}) \subset \{x + \sqrt{-1}y \mid y > C\}$ and $\psi(S_{r,r_4}) \subset \{x + \sqrt{-1}y \mid x < -C\}$. In particular, we obtain $\text{Im}(\psi) \cap \{\max(|x|, |y|) \leq C\} = \emptyset$.

We consider the case $P \notin \text{Im}(\psi)$. Then, applying the maximum principle to $(z - P)^{-1} \circ \psi$, we obtain that $\text{Im} \psi \cap \{z_1 \mid |P - z_1| \leq 80C\} = \emptyset$. In particular, we have $\text{Im} \psi \cap \{x + \sqrt{-1}y \mid \max(|x|, |y|) \leq C\} = \emptyset$. Then, the claim of Lemma 3.22 follows. \blacksquare

By applying the same argument to any hypersurface H , we obtain neighbourhoods of H

$$H \subset V_{1,H} \Subset V_{2,H} \Subset U_H$$

with the following property:

- For any $r < R_0$, if $\varphi(S_r) \cap V_{1,H} \neq \emptyset$, then $\varphi(S_r) \subset V_{2,H}$.
- For any $r_1 < r_2 < R_0$, if $\varphi(S_{r_1}), \varphi(S_{r_2}) \subset V_{2,H}$, then $\varphi(S_{r_1,r_2}) \subset U_H$.

Let us return to the proof of Theorem 3.21. Let P be any point of \mathbb{P}^n . We can take H_1, \dots, H_n such that $\{P\} = \bigcap_{j=1}^n H_j$. We put $V_{1,P} := \bigcap_{j=1}^n V_{1,H_j}$, $V_{2,P} := \bigcap_{j=1}^n V_{2,H_j}$ and $U_P := \bigcap_{j=1}^n U_{H_j}$. Then, the following holds:

- Let $r < R_0$. If $\varphi(S_r) \cap V_{1,P} \neq \emptyset$, then $\varphi(S_r) \subset V_{2,P} \subset U_P$.
- Let $0 < r_1 < r_2 < R_0$. If $\varphi(S_{r_1}), \varphi(S_{r_2}) \subset V_{2,P}$, then $\varphi(S_{r_1,r_2}) \subset U_P$.

Because \mathbb{P}^n is compact, we can find $P_1, \dots, P_N \in \mathbb{P}^n$ such that $\mathbb{P}^n = \bigcup_{j=1}^N V_{1,P_j}$. We immediately obtain Theorem 3.21 from the following lemma.

Lemma 3.23 *There exist $r_0 > 0$ and P_j such that $\varphi(S_r) \subset U_{P_j}$ for any $r \leq r_0$.*

Proof We have $\Delta^* = \bigcup_{j=1}^N \varphi^{-1}(V_{1,P_j})$. There exist P_j and an infinite sequence $w_i \in \varphi^{-1}(V_{1,P_j})$ ($i = 1, 2, \dots$) such that (i) $w_i \rightarrow 0$, (ii) $|w_i| > |w_{i+1}|$. Then, we have $\varphi(S_{|w_i|}) \subset V_{2,P_j}$, and hence $\varphi(S_{|w_{i+1}|, |w_i|}) \subset U_{P_j}$. Then, the claim of Lemma 3.23 follows. \blacksquare

3.6 Asymptotic harmonic bundles

In this subsection, we explain that some of the results for the asymptotic behaviour of wild harmonic bundles are naturally extended for Higgs bundles with a hermitian metric satisfying the Hitchin equation up to an exponentially small term. It is used in the proof of Theorem 3.17.

We put $X := \Delta_z = \{z \in \mathbb{C} \mid |z| < 1\}$, $\overline{X} := \{|z| \leq 1\}$, and $D := \{0\}$. Let $g_{\mathbf{p}}$ be the Poincaré metric of $X \setminus D$. Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle on $\overline{X} \setminus D$. We suppose that there exists a decomposition

$$(E, \theta) = \bigoplus_{\substack{\mathbf{a} \in z^{-1}\mathbb{C}[z^{-1}] \\ \alpha \in \mathbb{C}}} (E_{\mathbf{a},\alpha}, \theta_{\mathbf{a},\alpha}) \quad (35)$$

such that, for the expression $\theta_{\mathbf{a},\alpha} = d\mathbf{a} + \alpha dz/z + f_{\mathbf{a},\alpha} dz/z$, the eigenvalues of $f_{\mathbf{a},\alpha}(z)$ goes to 0 when $z \rightarrow 0$. We put $\text{Irr}(\theta) := \{\mathbf{a} \mid \exists \alpha \text{ such that } E_{\mathbf{a},\alpha} \neq 0\}$.

For any $\mathbf{a}(z) = \sum_{j \geq -N} \mathbf{a}_j z^j$ with $\mathbf{a}_{-N} \neq 0$, we set $\text{ord}(\mathbf{a}) := -N$. We also set $\text{ord}(0) := 0$. We take a negative number p satisfying $p < \min\{\text{ord}(\mathbf{a} - \mathbf{b}) \mid \mathbf{a}, \mathbf{b} \in \text{Irr}(\theta), \mathbf{a} \neq \mathbf{b}\}$.

Let h be a hermitian metric of E . Let θ^\dagger denote the adjoint of θ with respect to h . Let $F(h)$ denote the curvature of $(E, \overline{\partial}_E, h)$. We impose the following condition for some $C_0 > 0$ and $\epsilon_0 > 0$:

$$\left| F(h) + [\theta, \theta^\dagger] \right|_{h, g_{\mathbf{p}}} \leq C_0 \exp(-\epsilon_0 |z|^p) \quad (36)$$

3.6.1 Asymptotic orthogonality and acceptability

We have the following version of Simpson's main estimate.

Proposition 3.24 *Suppose that $(E, \bar{\partial}_E, \theta, h)$ satisfies (36).*

- If $\mathfrak{a} \neq \mathfrak{b}$, there exists $\epsilon > 0$ such that $E_{\mathfrak{a}, \alpha}$ and $E_{\mathfrak{b}, \beta}$ are $O(\exp(-\epsilon|z|^{\text{ord}(\mathfrak{a}-\mathfrak{b})}))$ -asymptotically orthogonal, i.e., there exists $C > 0$ such that, for any $u, v \in E|_Q$, we have $|h(u, v)| \leq C_1 \exp(-\epsilon|z(Q)|^{\text{ord}(\mathfrak{a}-\mathfrak{b})})$.
- If $\alpha \neq \beta$, there exists $\epsilon > 0$ such that $E_{\mathfrak{a}, \alpha}$ and $E_{\mathfrak{a}, \beta}$ are $O(|z|^\epsilon)$ -asymptotically orthogonal.
- $\theta_{\mathfrak{a}, \alpha} - (d\mathfrak{a} + \alpha dz/z) \text{id}_{E_{\mathfrak{a}, \alpha}}$ is bounded with respect to h and the Poincaré metric $g_{\mathbf{P}}$.

Proof By considering the tensor product with a harmonic bundle of a rank one, we may assume $p < \min\{\text{ord}(\mathfrak{a}) \mid \mathfrak{a} \in \text{Irr}(\theta)\}$. We have a map $\eta_\ell : z^{-1}\mathbb{C}[z^{-1}] \rightarrow \mathcal{I}_\ell := z^{-\ell}\mathbb{C}[z^{-1}]$ by forgetting the terms $\sum_{j \geq -\ell+1} \mathfrak{a}_j z^j$. For each $\mathfrak{b} \in \mathcal{I}_\ell$, we set $E_{\mathfrak{b}}^{(\ell)} := \bigoplus_{\eta_\ell(\mathfrak{a})=\mathfrak{b}} \bigoplus_{\alpha \in \mathbb{C}} E_{\mathfrak{a}, \alpha}$. Let $\pi_{\mathfrak{a}}^{(\ell)}$ denote the projection of E onto $E_{\mathfrak{b}}^{(\ell)}$ with respect to the decomposition $E = \bigoplus_{\mathfrak{b}} E_{\mathfrak{b}}^{(\ell)}$. In the case $\ell = 1$, we omit the superscript (1).

Let $\text{Irr}(\theta, \ell)$ be the image of $\text{Irr}(\theta)$ by η_ℓ . We take a total order \leq' on $\text{Irr}(\theta, \ell)$ for each ℓ such that the induced map $\text{Irr}(\theta, 1) \rightarrow \text{Irr}(\theta, \ell)$ is order-preserving. Let $E_{\mathfrak{b}}'^{(\ell)}$ be the orthogonal complement of $\bigoplus_{\mathfrak{c} <'_\ell \mathfrak{b}} E_{\mathfrak{c}}$ in $\bigoplus_{\mathfrak{c} \leq'_\ell \mathfrak{b}} E_{\mathfrak{c}}$. Let $\pi_{\mathfrak{b}}'^{(\ell)}$ be the orthogonal projection onto $E_{\mathfrak{b}}'^{(\ell)}$. In the case $\ell = 1$, we omit the superscript (1). We have $\pi_{\mathfrak{b}}'^{(\ell)} = \sum_{\eta_\ell(\mathfrak{a})=\mathfrak{b}} \pi_{\mathfrak{a}}'$.

We put $\zeta_\ell := \eta_\ell - \eta_{\ell+1}$. We have the expression $\theta = f dz$. We put $f^{(\ell)} := f - \sum_{\mathfrak{a}} \partial_z \eta_{\ell+1}(\mathfrak{a}) \pi_{\mathfrak{a}}$, $\mu^{(\ell)} := f^{(\ell)} - \sum_{\mathfrak{a}} \partial_z \zeta_\ell(\mathfrak{a}) \pi_{\mathfrak{a}}'$ and $\mathcal{R}_{\mathfrak{b}}^{(\ell)} := \pi_{\mathfrak{b}}^{(\ell)} - \pi_{\mathfrak{b}}'^{(\ell)}$. We consider the following claims.

$$(P_\ell) \quad |f^{(\ell')}|_h = O(|z|^{-\ell'-1}) \text{ for } \ell' \geq \ell.$$

$$(Q_\ell) \quad |\mu^{(\ell')}|_h = O(|z|^{-\ell'}) \text{ for } \ell' \geq \ell.$$

$$(R_\ell) \quad |\mathcal{R}_{\mathfrak{b}}^{(\ell')}|_h = O(\exp(-C|z|^{-\ell'})) \text{ for } \ell' \geq \ell \text{ and for } \mathfrak{b} \in \text{Irr}(\theta, \ell').$$

The asymptotic orthogonality of $E_{\mathfrak{a}, \alpha}$ and $E_{\mathfrak{b}, \beta}$ ($\mathfrak{a} \neq \mathfrak{b}$) follows from (R_1) .

In the proof of Theorem 7.2.1 of [38], we proved the claims for a wild harmonic bundle by using a descending induction on ℓ . The essentially same argument can work. We give an indication for a modification in this situation.

We have the expression $\theta^\dagger = f^\dagger d\bar{z}$. Let $\Delta := -\partial_z \partial_{\bar{z}}$. If a holomorphic section s of $\text{End}(E)$ satisfies $[s, f] = 0$, we obtain the following inequality from (36):

$$\Delta \log |s|_h^2 \leq -\frac{|[f^\dagger, s]|_h^2}{|s|_h^2} + C_0 \exp(-\epsilon_0 |z|^p) \quad (37)$$

Let $f^{(\ell)\dagger}$ denote the adjoint of $f^{(\ell)}$ with respect to h . Suppose $P_{\ell+1}$, $Q_{\ell+1}$ and $R_{\ell+1}$. By applying (37) to $f^{(\ell)}$, we obtain the following, as in (99) of [38]:

$$\Delta \log |f^{(\ell)}|_h^2 \leq -\frac{|[f^{(\ell)\dagger}, f^{(\ell)}]|_h^2}{|f^{(\ell)}|_h^2} + C_1$$

Then, by the same argument as that in §7.3.2–§7.3.3 of [38], we obtain P_ℓ and Q_ℓ . We put

$$k_{\mathfrak{b}}^{(\ell)} := \log(|\pi_{\mathfrak{b}}^{(\ell)}|_h^2 / |\pi_{\mathfrak{b}}'^{(\ell)}|_h^2) = \log(1 + |\mathcal{R}_{\mathfrak{b}}^{(\ell)}|_h^2 / |\pi_{\mathfrak{b}}'^{(\ell)}|_h^2).$$

By applying (37) to $\pi_{\mathfrak{b}}^{(\ell)}$, we obtain

$$\Delta \log k_{\mathfrak{b}}^{(\ell)} \leq -\frac{|[f^\dagger, \pi_{\mathfrak{b}}^{(\ell)}]|_h^2}{|\pi_{\mathfrak{b}}^{(\ell)}|_h^2} + C_0 \exp(-\epsilon_0 |z|^p).$$

There exists $C_1 > 0$ and $R_1 > 0$ such that the following holds for any $|z| < R_1$:

$$\begin{aligned} \Delta \exp(-A|z|^{-\ell}) &\geq -\exp(-A|z|^{-\ell}) \left(\frac{\ell^2}{4} A^2 |z|^{-2(\ell+1)} \right) \\ &\geq -\exp(-A|z|^{-\ell}) \frac{\ell^2}{4} A^2 C_1 |z|^{-2(\ell+1)} + C_0 \exp(-\epsilon_0 |z|^p) \end{aligned} \quad (38)$$

Hence, we obtain R_ℓ by using the argument in §7.3.4 of [38]. Similarly, we obtain the asymptotic orthogonality of $E_{\mathbf{a},\alpha}$ and $E_{\mathbf{a},\beta}$ ($\alpha \neq \beta$), and the boundedness of $\theta_{\mathbf{a},\alpha} - (d\mathbf{a} + \alpha dz/z) \text{id}_{E_{\mathbf{a},\alpha}}$ by using the argument in §7.3.5–§7.3.7 of [38] with (37). \blacksquare

We obtain the following corollary. (See §7.2.5 of [38] for the argument.)

Corollary 3.25 *($E, \bar{\partial}_E, h$) is acceptable, i.e., the curvature $F(h)$ is bounded with respect to h and $g_{\mathbf{P}}$.* \blacksquare

3.6.2 Prolongation and the norm estimate

For any $U \subset X$ and for any $a \in \mathbb{R}$, let $\mathcal{P}_a E(U)$ denote the space of holomorphic sections s of $E(U \setminus D)$ such that $|s|_h = O(|z|^{-a-\epsilon})$ ($\forall \epsilon$) locally around any point of U . According to a general theory of acceptable bundles, we obtain a locally free \mathcal{O}_X -module $\mathcal{P}_a E$, and a filtered bundle $\mathcal{P}_* E = (\mathcal{P}_a E \mid a \in \mathbb{R})$. (See §5.5 for a review of filtered bundles.) The decomposition (35) is extended to a decomposition of $\mathcal{P}_a E$:

$$\mathcal{P}_a E = \bigoplus \mathcal{P}_a E_{\mathbf{a},\alpha}$$

We set $\mathcal{P}E := \bigcup \mathcal{P}_a E$ and $\mathcal{P}E_{\mathbf{a},\alpha} := \bigcup \mathcal{P}_a E_{\mathbf{a},\alpha}$. We set $\text{Gr}_a^{\mathcal{P}}(E) := \mathcal{P}_a E / \mathcal{P}_{<a} E$, which we naturally regard as a \mathbb{C} -vector space.

By Proposition 3.24, θ gives a section of $\text{End}(\mathcal{P}E) \otimes \Omega_X^1$, which preserves the decomposition $\mathcal{P}E = \bigoplus \mathcal{P}E_{\mathbf{a},\alpha}$. By the estimate in Proposition 3.24, $\theta_{\mathbf{a},\alpha} - (d\mathbf{a} + \alpha dz/z) \text{id}_{E_{\mathbf{a},\alpha}}$ is logarithmic with respect to the lattice $\mathcal{P}_a E_{\mathbf{a},\alpha}$. Hence, we have the induced endomorphism $\text{Res}(\theta_{\mathbf{a},\alpha})$ of $\text{Gr}_a^{\mathcal{P}} E_{\mathbf{a},\alpha}$, which has a unique eigenvalue α . We set $\text{Res}(\theta) = \bigoplus \text{Res}(\theta_{\mathbf{a},\alpha})$. Let $W \text{Gr}_a^{\mathcal{P}}(E)$ be the monodromy weight filtration of the nilpotent part of $\text{Res}(\theta)$.

For each section s of $\mathcal{P}E$, let $\deg^{\mathcal{P}}(s) := \min\{a \mid s \in \mathcal{P}_a E\}$. For any $g \in \text{Gr}_a^{\mathcal{P}} E$, let $\deg^W(g) := \min\{m \mid g \in W_m\}$. Let $\mathbf{v} = (v_i)$ be a frame of $\mathcal{P}_a E$ which is compatible with the decomposition $\mathcal{P}_a E = \bigoplus \mathcal{P}_a E_{\mathbf{a},\alpha}$, the parabolic filtration and the weight filtration, i.e., for any $a - 1 < b \leq a$, each v_i is a section of a direct summand $E_{\mathbf{a},\alpha}$, the tuple $\mathbf{v}^{(b)} := (v_i \mid \deg^{\mathcal{P}} v_i = b)$ induces a base $[\mathbf{v}^{(b)}] := ([v_i^{(b)}])$ of $\text{Gr}_b^{\mathcal{P}} E$, and the tuple $[\mathbf{v}^{(b),m}] := ([v_i^{(b)}] \mid \deg^W v_i^{(b)} = m)$ induces a base of $\text{Gr}_m^W \text{Gr}_b^{\mathcal{P}} E$. We set $a_i := \deg^{\mathcal{P}}(v_i)$ and $k_i := \deg^W(v_i)$. Let h_0 be the metric of E determined by $h_0(v_i, v_i) = |z|^{2a_i} (-\log |z|)^{k_i}$ and $h_0(v_i, v_j) = 0$ ($i \neq j$). The following proposition can be shown by the argument in §8.1.2 of [38].

Proposition 3.26 *h and h_0 are mutually bounded.* \blacksquare

3.6.3 Connection form

Let \mathbf{v} be a frame of $\mathcal{P}_a E$, which is compatible with the decomposition $\mathcal{P}_a E = \bigoplus \mathcal{P}_a E_{\mathbf{a},\alpha}$, the parabolic filtration and the weight filtration. Let G be the endomorphism of E determined by $G(v_i) dz = \partial v_i$ for $i = 1, \dots, \text{rank } E$. We can show the following by the arguments of Lemma 7.5.5, Lemma 10.1.3 and Proposition 10.3.3 of [38].

Lemma 3.27 *We have $|G|_h = O(|z|^{-1})$. For the decomposition $G = \sum G_{(\mathbf{a},\alpha),(\mathbf{b},\beta)}$ according to $E = \bigoplus E_{\mathbf{a},\alpha}$, we have the following estimate for some $\epsilon > 0$:*

$$|G_{(\mathbf{a},\alpha),(\mathbf{b},\beta)}|_h = \begin{cases} O(\exp(-\epsilon |z|^{\text{ord}(\mathbf{a}-\mathbf{b})})) & (\mathbf{a} \neq \mathbf{b}) \\ |z|^{-1+\epsilon} & (\mathbf{a} = \mathbf{b}, \alpha \neq \beta) \end{cases}$$

We have the expression $\theta = f dz$. Let us consider $\partial_h f$. Let Θ be determined by $f\mathbf{v} = \mathbf{v}\Theta$. Let C be determined by $\partial_h \mathbf{v} = \mathbf{v}C$. We have $(\partial_h f)\mathbf{v} = \mathbf{v}(\partial_z \Theta + [C, \Theta])$ and $[G, f]\mathbf{v} = \mathbf{v}[C, \Theta]$. We have the decompositions $\partial_h f = \sum (\partial_h f)_{(\mathbf{a}, \alpha), (\mathbf{b}, \beta)}$ and $\bar{\partial} f^\dagger = \sum (\bar{\partial} f^\dagger)_{(\mathbf{a}, \alpha), (\mathbf{b}, \beta)}$ according to $E = \bigoplus E_{\mathbf{a}, \alpha}$.

Corollary 3.28 *Let $m := \min\{\text{ord}(\mathbf{a}) \mid \mathbf{a} \in \text{Irr}(\theta)\}$. If $m < 0$, we have $\partial_{E, h} f = O(|z|^{-2+m} dz)$ with respect to h and $dz d\bar{z}$. We have*

$$|(\partial_{E, h} f)_{(\mathbf{a}, \alpha), (\mathbf{b}, \beta)}|_h = \begin{cases} O(\exp(-\epsilon |z|^{\text{ord}(\mathbf{a}-\mathbf{b})})) & (\mathbf{a} \neq \mathbf{b}) \\ |z|^{\epsilon-2} & (\mathbf{a} = \mathbf{b}, \alpha \neq \beta) \end{cases}$$

We also have the following:

$$|(\bar{\partial}_E f^\dagger)_{(\mathbf{a}, \alpha), (\mathbf{b}, \beta)}|_h = \begin{cases} O(\exp(-\epsilon |z|^{\text{ord}(\mathbf{a}-\mathbf{b})})) & (\mathbf{a} \neq \mathbf{b}) \\ |z|^{\epsilon-2} & (\mathbf{a} = \mathbf{b}, \alpha \neq \beta) \end{cases}$$

Proof It follows from Lemma 3.27. ■

3.6.4 Some estimate

Let t be a C^∞ -endomorphism of E . According to the decomposition $E = \bigoplus E_{\mathbf{a}, \alpha}$, we have the decomposition $t = \sum t_{(\mathbf{a}, \alpha), (\mathbf{b}, \beta)}$, where $t_{(\mathbf{a}, \alpha), (\mathbf{b}, \beta)} \in \text{Hom}(E_{\mathbf{b}, \beta}, E_{\mathbf{a}, \alpha})$. Let \mathcal{C} be the set of C^∞ -endomorphisms t such that the following holds for some $\epsilon > 0$ which may depend on t :

$$|t_{(\mathbf{a}, \alpha), (\mathbf{b}, \beta)}|_h = \begin{cases} O(|z|^\epsilon \exp(-\epsilon |z|^{\text{ord}(\mathbf{a}-\mathbf{b})})) & ((\mathbf{a}, \alpha) \neq (\mathbf{b}, \beta)) \\ O(1) & (\text{otherwise}) \end{cases}$$

Note that \mathcal{C} is closed under the addition and the composition.

Proposition 3.29 *Suppose t and $|z|^2 \partial_z \partial_{\bar{z}} t$ are contained in \mathcal{C} . Then, $z \partial_z t$ and $\bar{z} \partial_{\bar{z}} t$ are also contained in \mathcal{C} .*

Proof Let $\Psi : \mathbb{H} := \{u \in \mathbb{C} \mid \text{Im } u > 0\} \longrightarrow \{|z| < 1\}$ be given by $\Psi(u) = \exp(u)$. Because $\Psi^* t$ and $\partial_u \partial_{\bar{u}} \Psi^*(t)$ are bounded, we obtain that $\partial_u \Psi^* t$ and $\partial_{\bar{u}} \Psi^* t$ are also bounded.

In the following argument, positive constants ϵ can change. We use the notation in the proof of Proposition 3.24. We clearly have $\partial_{\bar{z}} \pi_{\mathbf{b}}^{(\ell)} = 0$. We have $\partial_z \pi_{\mathbf{b}}^{(\ell)} = O(\exp(-\epsilon |z|^{-\ell}))$ by Lemma 3.27. We also have $\partial_{\bar{z}} \partial_z \pi_{\mathbf{b}}^{(\ell)} = [F(h), \pi_{\mathbf{b}}^{(\ell)}] = O(\exp(-\epsilon |z|^{-\ell}))$.

We have the decomposition $t = \sum t_{\mathbf{a}, \mathbf{b}}^{(\ell)}$ according to the decomposition $E = \bigoplus E_{\mathbf{a}}^{(\ell)}$. We have $t_{\mathbf{a}, \mathbf{b}}^{(\ell)} = O(\exp(-\epsilon |z|^{-\ell}))$ if $\mathbf{a} \neq \mathbf{b}$. Hence, we have

$$[t, \pi_{\mathbf{b}}^{(\ell)}] = \sum_{\mathbf{a} \neq \mathbf{b}} t_{\mathbf{a}, \mathbf{b}}^{(\ell)} - \sum_{\mathbf{a} \neq \mathbf{b}} t_{\mathbf{b}, \mathbf{a}}^{(\ell)} = O(\exp(-\epsilon |z|^{-\ell}))$$

We also have $|z|^2 \partial_{\bar{z}} \partial_z [t, \pi_{\mathbf{b}}^{(\ell)}] = [|z|^2 \partial_{\bar{z}} \partial_z t, \pi_{\mathbf{b}}^{(\ell)}] + [\bar{z} \partial_{\bar{z}} t, z \partial_z \pi_{\mathbf{b}}^{(\ell)}] + [t, |z|^2 \partial_{\bar{z}} \partial_z \pi_{\mathbf{b}}^{(\ell)}] = O(\exp(-\epsilon |z|^{-\ell}))$. Hence, we obtain $z \partial_z [t, \pi_{\mathbf{b}}^{(\ell)}] = O(\exp(-\epsilon |z|^{-\ell}))$ and $\bar{z} \partial_{\bar{z}} [t, \pi_{\mathbf{b}}^{(\ell)}] = O(\exp(-\epsilon |z|^{-\ell}))$. Therefore, we obtain $z \partial_z t_{\mathbf{a}, \mathbf{b}}^{(\ell)} = O(\exp(-\epsilon |z|^{-\ell}))$ and $\bar{z} \partial_{\bar{z}} t_{\mathbf{a}, \mathbf{b}}^{(\ell)} = O(\exp(-\epsilon |z|^{-\ell}))$ for $\mathbf{a} \neq \mathbf{b}$.

We have $z \partial_z \pi_{\mathbf{a}, \alpha} = O(|z|^\epsilon)$ and $|z|^2 \partial_{\bar{z}} \partial_z \pi_{\mathbf{a}, \alpha} = O(|z|^\epsilon)$ by Lemma 3.27. Then, we obtain $z \partial_z t_{(\mathbf{a}, \alpha), (\mathbf{b}, \beta)} = |z|^\epsilon$ and $\bar{z} \partial_{\bar{z}} t_{(\mathbf{a}, \alpha), (\mathbf{b}, \beta)} = |z|^\epsilon$ for $\alpha \neq \beta$. If $\mathbf{a} \neq \mathbf{b}$ with $\ell = \text{ord}(\mathbf{a} - \mathbf{b})$, we obtain the desired estimate by using $t_{(\mathbf{a}, \alpha), (\mathbf{b}, \beta)} = \pi_{\mathbf{a}, \alpha} \circ t_{\eta_\ell(\mathbf{a}), \eta_\ell(\mathbf{b})}^{(\ell)} \circ \pi_{\mathbf{b}, \beta}$. ■

3.6.5 Refined asymptotic orthogonality

We obtain an asymptotic orthogonality of the derivative by assuming the following with respect to h and $dz d\bar{z}$, in addition to (36):

$$\partial_{\bar{z}}\partial_z(F(h) + [\theta, \theta^\dagger]) = O(\exp(-\epsilon_0|z|^p)). \quad (39)$$

Let \mathbf{v} be a frame of holomorphic \mathcal{P}_0E , compatible with the decomposition $\mathcal{P}_0E = \bigoplus \mathcal{P}_0E_{\mathbf{a}, \alpha}$, the parabolic filtration and the weight filtration. Let (\mathbf{a}_i, α_i) be determined by $v_i \in \mathcal{P}_0E_{\mathbf{a}_i, \alpha_i}$. We say that a matrix valued function $B = (B_{ij})$ satisfies the condition \mathcal{C}_1 , if the following holds for some $\epsilon > 0$ which may depend on B :

$$B_{ij} = \begin{cases} O(|z|^\epsilon \exp(-\epsilon|z|^{\text{ord}(\mathbf{a}_i - \mathbf{a}_j))) & ((\mathbf{a}_i, \alpha_i) \neq (\mathbf{a}_j, \alpha_j)) \\ O(|v_i|_h |v_j|_h) & (\text{otherwise}) \end{cases}$$

Let H be the matrix valued function determined by $H_{ij} = h(v_i, v_j)$. Lemma 3.27 implies that $z\partial_z H$ and $\bar{z}\partial_{\bar{z}} H$ satisfy the condition \mathcal{C}_1 .

Proposition 3.30 $(|z|^2 \partial_{\bar{z}} \partial_z)^2 H$ satisfies the condition \mathcal{C}_1 .

Proof Let $G(A)$ denote the endomorphism determined by \mathbf{v} and a matrix-valued function A . By Lemma 3.27, we have

$$G(H^{-1} z \partial_z H), G(H^{-1} \bar{z} \partial_{\bar{z}} H), G(\bar{H}^{-1} z \partial_z \bar{H}), G(\bar{H}^{-1} \bar{z} \partial_{\bar{z}} \bar{H}) \in \mathcal{C}.$$

Because $G(\bar{z} \partial_{\bar{z}} (\bar{H}^{-1} z \partial_z \bar{H})) = |z|^2 F(h) \in \mathcal{C}$, we have $G(\bar{H}^{-1} |z|^2 \partial_{\bar{z}} \partial_z \bar{H}) \in \mathcal{C}$.

We have the expression $\theta = f dz$. We have $\partial_{\bar{z}} \partial_z [f, f^\dagger] = [[F(h) \bar{z}, z], f], f^\dagger] + [\partial_z f, \bar{\partial}_z f^\dagger]$. It gives an estimate for $\partial_{\bar{z}} \partial_z [f, f^\dagger]$ by Corollary 3.28, from which we can deduce that $|z|^2 \partial_z \partial_{\bar{z}} (|z|^2 F(h)) \in \mathcal{C}$. By Proposition 3.29, we obtain $z \partial_z (|z|^2 F(h)) \in \mathcal{C}$ and $\bar{z} \partial_{\bar{z}} (|z|^2 F(h)) \in \mathcal{C}$. We obtain $G(\bar{z} \partial_{\bar{z}} (\bar{z} \partial_{\bar{z}} (\bar{H}^{-1} z \partial_z \bar{H}))), G(z \partial_z (z \partial_z (\bar{H}^{-1} z \partial_z \bar{H}))) \in \mathcal{C}$. We obtain $G(\bar{H}^{-1} (\bar{z} \partial_{\bar{z}})^2 z \partial_z \bar{H}), G(\bar{H}^{-1} z \partial_z (\bar{z} \partial_{\bar{z}})^2 \bar{H}) \in \mathcal{C}$. Then, we obtain $G(\bar{H}^{-1} (z \partial_z)^2 (\bar{z} \partial_{\bar{z}}^2 \bar{H})) \in \mathcal{C}$ from $|z|^2 \partial_z \partial_{\bar{z}} (|z|^2 F(h)) \in \mathcal{C}$. It implies the claim of the lemma. \blacksquare

Corollary 3.31 $(z \partial_z)^2 H$ satisfies the condition \mathcal{C}_1 . \blacksquare

Remark 3.32 The estimate as in Corollary 3.31 will be used in the study for the extension of the associated twistor family, which will be discussed elsewhere. \blacksquare

4 L^2 -instantons on $T \times \mathbb{C}$

4.1 Some standard property

4.1.1 Instantons of rank one

Let (E, ∇, h) be an L^2 -instanton on $T \times \mathbb{C}$ with $\text{rank } E = 1$.

Lemma 4.1 (E, ∇, h) is a unitary flat bundle.

Proof Because $\text{rank } E = 1$, we have $(\nabla_z \nabla_{\bar{z}} + \nabla_w \nabla_{\bar{w}}) F_{z\bar{z}} = 0$ and $(\nabla_z \nabla_{\bar{z}} + \nabla_w \nabla_{\bar{w}}) F_{z\bar{w}} = 0$. We obtain the following inequalities:

$$-(\partial_w \partial_{\bar{w}} + \partial_z \partial_{\bar{z}}) |F_{z\bar{z}}|^2 \leq 0, \quad -(\partial_w \partial_{\bar{w}} + \partial_z \partial_{\bar{z}}) |F_{z\bar{w}}|^2 \leq 0.$$

By applying the fiber integral for $T \times \mathbb{C} \rightarrow \mathbb{C}$, we obtain $-\partial_w \partial_{\bar{w}} \|F_{z\bar{z}}\|^2 \leq 0$ and $-\partial_w \partial_{\bar{w}} \|F_{z\bar{w}}\|^2 \leq 0$. Because the functions $\|F_{z\bar{z}}\|^2$ and $\|F_{z\bar{w}}\|^2$ are L^1 on \mathbb{C}_w , they are 0. \blacksquare

Corollary 4.2 Let (E, ∇, h) be an L^2 -instanton on $T \times \mathbb{C}$ of an arbitrary rank. Then, $\det(E, \nabla, h)$ is a flat unitary bundle, i.e., we have $\text{Tr } F(\nabla) = 0$. \blacksquare

If we do not impose the L^2 -property, there exist much more instantons of rank one on $T \times \mathbb{C}$.

(i) Let \mathbf{a} be any holomorphic function on \mathbb{C} . Then, the trivial holomorphic line bundle $\mathcal{O}_{\mathbb{C}}$ with the trivial metric and the Higgs field $d\mathbf{a}$ gives a harmonic bundle $\mathcal{L}(\mathbf{a})$ on \mathbb{C} . By the equivalence of Hitchin, we have the associated instanton on $T \times \mathbb{C}$.

(ii) Let ρ be an \mathbb{R} -valued harmonic function on $T \times \mathbb{C}$. Then, the trivial holomorphic line bundle $\mathcal{O}_{T \times \mathbb{C}}$ with the metric h_ρ given by $\log h_\rho(e, e) = \rho$ gives an instanton $\mathcal{L}(\rho)$ on $T \times \mathbb{C}$. Note that there exist many harmonic functions which is not the real part of a holomorphic function on $T \times \mathbb{C}$. We can construct such a function by using a Bessel function $I_0(r) = \int_{-1}^1 \cosh(rt)(t^2 - 1)^{-1/2} dt$ which satisfies $I_0'' + r^{-1}I_0' - I_0 = 0$. It is a C^∞ -function on \mathbb{R} , satisfying $I_0(r) = I_0(-r)$. In particular, $\kappa(w) := I_0(|w|)$ gives a C^∞ -function on \mathbb{C} satisfying $(-\partial_w \partial_{\bar{w}} + 4)\kappa = 0$. By using the Fourier expansion on $T \times \mathbb{C}$ in a standard way, we can construct a harmonic function ρ on $T \times \mathbb{C}$ from κ such that ρ is not constant along T . (See [27].) It is not the real part of any holomorphic function.

In general, any instanton of rank one $(E, \bar{\partial}_E, h)$ can be expressed as the tensor product of instantons of types (i) and (ii). Indeed, by considering the support $\text{RFM}_-(E, \bar{\partial}_E)$, we obtain a holomorphic function $\mathbb{C} \rightarrow T^\vee$. Because \mathbb{C} is simply connected, it is lifted to a holomorphic function $\mathbf{b} : \mathbb{C} \rightarrow \mathbb{C}$. We have a holomorphic function \mathbf{a} such that $\partial_w \mathbf{a} = \mathbf{b}$. Then, we can observe that $(E, \bar{\partial}_E, h)$ is isomorphic to $\mathcal{L}(\mathbf{a}) \otimes \mathcal{L}(\rho)$ for a harmonic function ρ on $T \times \mathbb{C}$.

4.1.2 Polystability of the associated filtered bundle

Let (E, ∇, h) be an L^2 -instanton on $T \times \mathbb{C}$. Let $(E, \bar{\partial}_E)$ be the underlying holomorphic vector bundle on $T \times \mathbb{C}$. For a saturated $\mathcal{O}_{T \times \mathbb{C}}$ -subsheaf $\mathcal{F} \subset E$, let $h_{\mathcal{F}}$ denote the induced hermitian metric of the smooth part of \mathcal{F} . Let $F(h_{\mathcal{F}})$ denote the curvature. As in [8] and [45], we set

$$\deg(\mathcal{F}, h) := \sqrt{-1} \int_{T \times \mathbb{C}} \text{Tr}(\Lambda F(h_{\mathcal{F}})) \, \text{dvol}_{T \times \mathbb{C}}.$$

Let $\pi_{\mathcal{F}}$ denote the orthogonal projection of E to \mathcal{F} , where it is considered only on the smooth part of \mathcal{F} . By the Chern-Weil formula [45], we have

$$\deg(\mathcal{F}, h) = - \int_{T \times \mathbb{C}} |\bar{\partial} \pi|_h^2 \, \text{dvol}_{T \times \mathbb{C}}.$$

Lemma 4.3 *$\deg(\mathcal{F}, h)$ is finite, if and only if (i) the degree of $\mathcal{F}|_{T \times \{w\}}$ are 0 for any $w \in \mathbb{C}$, (ii) \mathcal{F} is extended to a subsheaf $\mathcal{P}_0 \mathcal{F}$ of $\mathcal{P}_0 E$. In that case, we have $\deg(\mathcal{F}, h) = \int_{z \times \mathbb{P}^1} \text{par-c}_1(\mathcal{P}_* \mathcal{F})$, where $\mathcal{P}_* \mathcal{F}$ denotes $\mathcal{P}_0 \mathcal{F}$ with the induced parabolic structure.*

Proof This type of claim is standard in the study of Kobayashi-Hitchin correspondence for parabolic objects, based on the fundamental results in [45] and [49]. We give only an indication in our situation by following [31].

By Lemma 10.6 of [45], $\mathcal{F}|_{\{z\} \times \mathbb{C}}$ is extended to a parabolic subsheaf, if $\int_{\mathbb{C}} |\bar{\partial} \pi|_{\{z\} \times \mathbb{C}}|^2 < \infty$. By the argument in the same lemma, we can show the converse, i.e., $\mathcal{F}|_{\{z\} \times \mathbb{C}}$ is extendable to a parabolic subsheaf $\mathcal{P}_* \mathcal{F}|_{\{z\} \times \mathbb{C}}$ of $\mathcal{P}_* E|_{\{z\} \times \mathbb{C}}$, if and only if it is L_1^2 -subbundle of $(E, h)|_{\{z\} \times \mathbb{C}}$. In that case, $\frac{\sqrt{-1}}{2\pi} \int_{\mathbb{C}} \text{Tr}(F(h_{\mathcal{F}}))|_{z \times \mathbb{C}}$ is equal to the parabolic degree by Lemma 10.5 of [45].

If the conditions (i) and (ii) are satisfied, then we have

$$\deg(\mathcal{F}, h) = \int_T \text{dvol}_T \left(\int_{z \times \mathbb{C}} \sqrt{-1} \text{Tr}(F(h_{\mathcal{F}})) \right) = 2\pi |T| \int_{z \times \mathbb{P}^1} \text{par-c}_1(\mathcal{P}_* \mathcal{F}) > -\infty.$$

Conversely, suppose $\deg(\mathcal{F}, h)$ is finite. Because

$$\deg(\mathcal{F}, h) \leq \int_{\mathbb{C}} \text{dvol}_{\mathbb{C}} \left(- \int_T |\nabla_{\bar{z}} \pi|^2 \, \text{dvol}_T \right) = 2\pi \int_{\mathbb{C}} \deg(\mathcal{F}|_{T \times \{w\}}) \, \text{dvol}_{\mathbb{C}},$$

we have $\deg(\mathcal{F}|_{T \times \{w\}}) = 0$ for any w . We obtain $-\deg(\mathcal{F}, h) = \int_T \text{dvol}_T \left(\int_{\mathbb{C}} |\bar{\partial} \pi|_{\{z\} \times \mathbb{C}}|^2 \right) < \infty$. Hence, there exists a thick subset $A \subset T^\vee$ such that $\mathcal{F}|_{z \times \mathbb{C}}$ is extendable for any $z \in A$. (A subset is thick, if it is not contained in a countable union of a complex analytically closed subsets.) Then, we obtain that \mathcal{F} is extendable according to Theorem 4.5 of [49]. \blacksquare

Corollary 4.4 \mathcal{P}_*E is polystable. We have $\deg(\mathcal{P}_*E) = 0$. (See §5.3.1 for the stability condition in this case.)

Proof The second claim directly follows from Lemma 4.3 and Corollary 4.2. Let $\mathcal{P}_*\mathcal{F}$ be a filtered subsheaf \mathcal{P}_*E satisfying (A1–2). Let \mathcal{F} be its restriction to $X \times \mathbb{C}$. By Lemma 4.3, we have $\mu(\mathcal{P}_*\mathcal{F}) = \mu(\mathcal{F}, h) \leq 0$. Moreover, if it is 0, the orthogonal projection onto \mathcal{F} is holomorphic. Hence, the orthogonal decomposition $E = \mathcal{F} \oplus \mathcal{F}^\perp$ is holomorphic. It is extended to a decomposition $\mathcal{P}_*E = \mathcal{P}_*\mathcal{F} \oplus \mathcal{P}_*\mathcal{F}^\perp$. Both \mathcal{F} and \mathcal{F}^\perp with the induced metrics are L^2 -instantons. Hence, we obtain the first claim of the corollary by an easy induction on the rank. \blacksquare

4.1.3 Uniqueness of the L^2 -instanton adapted to a filtered bundle

Let (E, ∇, h) be an L^2 -instanton on $T \times \mathbb{C}$. We have the associated filtered bundle \mathcal{P}_*E on $(T \times \mathbb{P}^1, T \times \{\infty\})$. Let h' be a hermitian metric of E , and let $\nabla_{h'}$ be a unitary connection of (E, h') such that (i) $(E, \nabla_{h'}, h')$ is an L^2 -instanton, (ii) the $(0, 1)$ -parts of $\nabla_{h'}$ and ∇_h are equal, (iii) h' is adapted to \mathcal{P}_*E .

Proposition 4.5 We have a holomorphic decomposition $(E, \bar{\partial}_E) = \bigoplus_i (E_i, \bar{\partial}_{E_i})$ such that (i) it is orthogonal with respect to both h and h' , (ii) for each i , there exists $\alpha_i > 0$ such that $h|_{E_i} = \alpha_i h'|_{E_i}$. In particular, we have $\nabla_h = \nabla_{h'}$.

Proof Let s be the self-adjoint endomorphism determined by $h' = h s$. According to [45], we have the following inequality (see p.876 of [45]):

$$-(\bar{\partial}_z \partial_z + \bar{\partial}_w \partial_w) \text{Tr}(s) + |\bar{\partial}(s) s^{-1/2}|_h^2 \leq 0$$

By taking the fiber integral for $T \times \mathbb{C} \rightarrow \mathbb{C}$, we obtain

$$-\partial_{\bar{w}} \partial_w \int_T \text{Tr}(s) + \int_T |\bar{\partial}(s) s^{-1/2}|_h^2 \leq 0$$

It implies that $\int_T \text{Tr}(s)$ is a subharmonic function on \mathbb{C}_w . By using the norm estimate for asymptotically harmonic bundle (Proposition 3.26), we obtain that h and h' are mutually bounded, i.e., s and s^{-1} are bounded with respect to both of h . Hence, we obtain that $\int_T \text{Tr}(s)$ is constant. We obtain $\int_T |\bar{\partial}(s) s^{-1/2}|_h^2 = 0$, which implies $\bar{\partial}(s) = 0$. Then, the claim of the proposition follows. \blacksquare

4.1.4 Instanton number

Let (E, ∇, h) be an L^2 -instanton on $T \times \mathbb{C}$. We have the associated filtered bundle \mathcal{P}_*E on $(T \times \mathbb{P}^1, T \times \{0\})$. Note that the second Chern class of $\mathcal{P}_a E$ is independent of $a \in \mathbb{R}$.

Proposition 4.6 We have the following equality:

$$\frac{1}{8\pi^2} \int_{T \times \mathbb{C}} \text{Tr}(F(h)^2) = \int_{T \times \mathbb{P}^1} c_2(\mathcal{P}_a E)$$

Proof Let $U \subset \mathbb{P}^1$ be a neighbourhood of ∞ . Let $\tau = w^{-1}$. Let (\mathcal{P}_*V, g) be a filtered bundle on $(U, 0)$ corresponding to \mathcal{P}_*E . We set $\theta := g dw$. We take a ramified covering $\varphi : (U', 0) \rightarrow (U, 0)$ given by $\tau = \xi^m$ such that we have the decomposition

$$\varphi^*(\mathcal{P}_*V, \theta) = \bigoplus_{\mathbf{a} \in \xi^{-1}\mathbb{C}[\xi^{-1}]} (\mathcal{P}_*V'_\mathbf{a}, \theta_\mathbf{a}), \quad (40)$$

where $\theta_\mathbf{a} - d\mathbf{a}$ are tame. We have the decomposition

$$\varphi^*\mathcal{P}_0V = \bigoplus_{\mathbf{a}} (\varphi^*\mathcal{P}_0E \cap \mathcal{P}V'_\mathbf{a}). \quad (41)$$

We take a frame \mathbf{v} of $\varphi^*\mathcal{P}_0V$ such that (i) it is compatible with the decomposition (41), (ii) for $\kappa \in \text{Gal}(\varphi)$, we have $\kappa^*v_i = a(\kappa, i)v_{b(\kappa, i)}$ for some $a(\kappa, i) \in \mathbb{C}^*$ with $|a(\kappa, i)| = 1$ and $1 \leq b(\kappa, i) \leq \text{rank } V$. Let h_0 be a hermitian

metric of φ^*V defined by $h_0(v_i, v_j) = \delta_{ij}$. Because it is $\text{Gal}(\varphi)$ -equivariant, it gives a hermitian metric of \mathcal{P}_0V . It induces a hermitian metric h_1 of $(E, \bar{\partial}_E)$. Let $\nabla_{h_1} = \bar{\partial}_E + \partial_{E, h_1}$ denote the Chern connection, and let $F(h_1)$ denote the curvature.

Let $\bar{\partial}_0$ denote the holomorphic structure of E obtained as the pull back of V . We have $\bar{\partial}_E = \bar{\partial}_0 + g d\bar{z}$. We have $\partial_{E, h} = \partial_{E, h} + g_h^\dagger dz$ and $\partial_{E, h_1} = \partial_{E, h_1} + g_{h_1}^\dagger dz$.

We have the expression $g\mathbf{v} = \mathbf{v}G$. Let \mathbf{u} denote the frame of E obtained as the pull back of \mathbf{v} . We have $\nabla_{h_1}\mathbf{u} = \mathbf{u}(G d\bar{z} + G_{h_0}^\dagger dz)$. We have $F(h_1)\mathbf{u} = \mathbf{u}(\partial_w G dw d\bar{z} + [G, G_{h_0}^\dagger] d\bar{z} dz)$. Hence, we have

$$|F(h_1)|_{h_1} = O(|w|^{-2} dw d\bar{z}) + O(|w|^{-1} dz d\bar{z}).$$

We have the expression $\nabla\mathbf{u} = \mathbf{u}(C dw + g d\bar{z} + g_h^\dagger dz)$. Let F_C be the endomorphism determined by $F_C\mathbf{u} = \mathbf{u}C$. We have

$$\nabla_h - \nabla_{h_1} = F_C dw + (g_h^\dagger - g_{h_1}^\dagger) dz.$$

According to Lemma 3.27, we have $|F_C|_h = O(|w|^{-1})$. It follows that $|F_C|_{h_1} = O(|w|^{-\rho_1})$ for some $\rho_1 > 0$. We have the expression

$$\varphi^*G = \bigoplus (\partial_w \mathbf{a}) I_{\mathbf{a}} + O(|w|^{-1}).$$

Here, $I_{\mathbf{a}}$ are diagonal matrices whose (i, i) -entry is 1 if $v_i \in \mathcal{P}V_{\mathbf{a}}'$, or 0 otherwise. Hence, by using Proposition 3.24, there exists $\rho_2 > 0$ such that $|g_h^\dagger - g_{h_1}^\dagger|_h = O(|w|^{-\rho_2})$ and $|g_h^\dagger - g_{h_1}^\dagger|_{h_1} = O(|w|^{-\rho_2})$. Hence, we have

$$\begin{aligned} F(h)(\nabla_h - \nabla_{h_1}) &= O(|w|^{-1-\rho_3} dw dz d\bar{z}) + O(|w|^{-1-\rho_3} dw dz d\bar{w}), \\ F(h_1)(\nabla_h - \nabla_{h_1}) &= O(|w|^{-1-\rho_3} dw dz d\bar{z}). \end{aligned}$$

Then, we obtain the following:

$$-\frac{1}{8\pi^2} \int_{T \times \mathbb{C}} \text{Tr}(F(h)^2) = -\frac{1}{8\pi^2} \int_{T \times \mathbb{P}^1} \text{Tr}(F(h_1)^2) = \int_{X \times \mathbb{P}^1} \text{ch}_2(\mathcal{P}_0 E) = - \int_{X \times \mathbb{P}^1} c_2(\mathcal{P}_0 E)$$

Thus, we are done. ■

4.2 Cohomology

Let (E, ∇, h) be an L^2 -instanton on $X := T \times \mathbb{C}$. The $(0, 1)$ -part of ∇ is denoted by $\bar{\partial}_E$. Let $\bar{X} := T \times \mathbb{P}^1$. We put $D := T \times \{\infty\}$. Let $A_c^{0,i}(E)$ denote the space of C^∞ -sections of $E \otimes \Omega^{0,i}$ on X with compact supports. Its cohomology group is denoted by $H_c^{0,i}(X, E)$. Let $A^{0,i}(\mathcal{P}_a E)$ denote the space of C^∞ -sections of $\mathcal{P}_a E \otimes \Omega^{0,i}$ on \bar{X} . Its cohomology group is $H^i(\bar{X}, \mathcal{P}_a E)$. In this subsection, we suppose that $0 \notin \mathcal{S}p_\infty(E)$.

Proposition 4.7 *The natural map $H_c^{0,i}(X, E) \longrightarrow H^{0,i}(\bar{X}, \mathcal{P}_a E)$ is an isomorphism for any $a \in \mathbb{R}$.*

Proof There exists $R > 0$ such that, if $|w| > R$, $E|_{T \times \{w\}}$ is semistable of degree 0, and $0 \notin \mathcal{S}p(E|_{T \times \{w\}})$. We have two consequences for a C^∞ -section s of $\mathcal{P}_a E$ on X_R .

- There exists a C^∞ -section t of $\mathcal{P}_a E$ on X_R such that $\nabla_{\bar{z}} t = s$.
- If $\nabla_{\bar{z}} s = 0$, then $s = 0$.

Then, the claim can be shown easily. ■

Let $A_{L^2}^{0,i}(E)$ be the space of L^2 -sections s of $E \otimes \Omega^{0,i}$ on X such that $\bar{\partial}_E s$ is also L^2 . The cohomology group of the complex $(A_{L^2}^{0,\bullet}(E), \bar{\partial}_E)$ is denoted by $H_{L^2}^{0,i}(X, E)$.

Proposition 4.8 *The natural map $H_c^{0,i}(X, E) \longrightarrow H_{L^2}^{0,i}(X, E)$ is an isomorphism.*

Proof Let $A_{L^2, c}^{0,i}(E) \subset A_{L^2}^{0,i}(E)$ be the subspace of the sections with compact supports. It gives a subcomplex, and its cohomology is denoted by $H_{L^2, c}^{0,i}(X, E)$.

Lemma 4.9 *The natural map $H_{L^2,c}^{0,i}(X, E) \longrightarrow H_{L^2}^{0,i}(X, E)$ is an isomorphism.*

Proof For any L^2 -section s of E on X_R , there exists an L^2 -section t of E on $X_{R'}$ ($R' > R$) such that $\nabla_{\bar{z}} t = s$ on $X_{R'}$. If an L^2 -section s of E on X_R satisfies $\nabla_{\bar{z}} s = 0$, then we have $s = 0$. Then, the claim of the lemma can be shown. \blacksquare

We take a smooth Kähler metric of \bar{X} . Let $B_{L^2}^{0,i}(\mathcal{P}_a E)$ be the space of L^2 -sections ω of $\mathcal{P}_a E$ on \bar{X} such that $\bar{\partial}\omega$ is L^2 . Let $B_{L^2,c}^{0,i}(\mathcal{P}_a E) \subset B_{L^2}^{0,i}(\mathcal{P}_a E)$ denote the subspace of the sections whose support is contained in X . By the same argument, the natural map $B_{L^2,c}^{0,\bullet}(\mathcal{P}_a E) \longrightarrow B_{L^2}^{0,\bullet}(\mathcal{P}_a E)$ is a quasi isomorphism. We have a natural identification $B_{L^2,c}^{0,\bullet}(\mathcal{P}_a E) = A_{L^2,c}^{0,\bullet}(E)$ as \mathbb{C} -linear spaces. By the L^2 -Dolbeault theorem, the cohomology group of $B_{L^2}^{0,\bullet}(\mathcal{P}_a E)$ is naturally isomorphic to $H^i(\bar{X}, \mathcal{P}_a E)$. Then, the claim of Proposition 4.8 follows. \blacksquare

Corollary 4.10 *$H_{L^2}^{0,i}(\bar{X}, E)$ is finite dimensional.* \blacksquare

Proposition 4.11 *We have $H^0(\bar{X}, \mathcal{P}_a E) = H^2(\bar{X}, \mathcal{P}_a E) = 0$.*

Proof Clearly $H^0(\bar{X}, \mathcal{P}_a(E)) = 0$. Let $p : \bar{X} \longrightarrow \mathbb{P}^1$ be the projection. We have $p_* E = 0$, and the support of $R^1 p_* E$ is 0-dimensional. Then, we obtain $H^2(\bar{X}, \mathcal{P}_a E) = 0$. \blacksquare

4.3 Exponential decay of harmonic forms

4.3.1 Statement

Let (E, ∇, h) be an L^2 -instanton on $T \times \mathbb{C}$. Let $\bar{\partial}_E$ denote the $(0, 1)$ -part of ∇ , and let $\bar{\partial}_E^*$ denote the formal adjoint with respect to h and $dz d\bar{z} + dw d\bar{w}$. We set $\Delta_E := \bar{\partial}_E^* \bar{\partial}_E + \bar{\partial}_E \bar{\partial}_E^*$.

Proposition 4.12 *Assume that $0 \notin Sp_\infty(E)$. Let ω be an L^2 -section of $E \otimes \Omega^{0,1}$ on $T \times \mathbb{C}$ such that $\Delta_E \omega = 0$. Then, we have $|\omega| = O(\exp(-C|w|))$ for some $C > 0$.*

4.3.2 An estimate

Take $R > 0$, and put $Y_R := \{|w| \geq R\}$ and $X_R := T \times Y_R$. Let (E, ∇, h) be an L^2 -instanton on X_R .

Lemma 4.13 *Assume that $0 \notin Sp_\infty(E)$. Suppose that ω is an L^2 -section of $E \otimes \Omega_{Y_R}^{0,1}$ such that $\bar{\partial}_E \omega = \bar{\partial}_E^* \omega = 0$. Then, there exists $C > 0$ such that $|\omega|_h = O(\exp(-C|w|))$.*

Proof Let $\omega = f d\bar{z} + g d\bar{w}$ be a harmonic form. We have $-\nabla_{\bar{w}} f + \nabla_{\bar{z}} g = 0$ and $\nabla_z f + \nabla_w g = 0$. We have the following equalities:

$$\begin{aligned} \nabla_w \nabla_{\bar{w}} f &= \nabla_w (\nabla_{\bar{z}} g) = \nabla_{\bar{z}} \nabla_w g + F_{w\bar{z}} g = -\nabla_{\bar{z}} \nabla_z f + F_{w\bar{z}} g = -\nabla_z \nabla_{\bar{z}} f + F_{z\bar{z}} f + F_{w\bar{z}} g \\ \nabla_w \nabla_{\bar{w}} g &= F_{w\bar{w}} g + \nabla_{\bar{w}} \nabla_w g = F_{w\bar{w}} g + \nabla_{\bar{w}} (-\nabla_z f) = F_{w\bar{w}} g - \nabla_z \nabla_{\bar{w}} f + F_{z\bar{w}} f = F_{w\bar{w}} g - \nabla_z \nabla_{\bar{z}} g + F_{z\bar{w}} f \end{aligned}$$

We obtain the following

$$\begin{aligned} -(\partial_w \bar{\partial}_{\bar{w}} + \partial_z \bar{\partial}_{\bar{z}})(f, f) &\leq -(\nabla_{\bar{z}} f, \nabla_{\bar{z}} f) - 2 \operatorname{Re}((\nabla_w \nabla_{\bar{w}} + \nabla_z \nabla_{\bar{z}})f, f) \\ &= -(\nabla_{\bar{z}} f, \nabla_{\bar{z}} f) - 2 \operatorname{Re}(F_{z\bar{z}} f + F_{z\bar{w}} g, f) \end{aligned} \quad (42)$$

Using the notation in §3.2.2, we obtain

$$-\partial_w \bar{\partial}_{\bar{w}} \|f\|^2 \leq -\|\nabla_{\bar{z}} f\|^2 + O(\|F\|(\|f\|^2 + \|g\|^2))$$

Similarly, we obtain

$$-\partial_w \bar{\partial}_{\bar{w}} \|g\|^2 \leq -\|\nabla_{\bar{z}} g\|^2 + O(\|F\|(\|f\|^2 + \|g\|^2))$$

By the assumption $0 \notin \mathcal{S}p_\infty(E)$, there exist $R_1 > R$ and $C_1 > 0$ such that, if $|w| \geq R_1$, we have $\|\partial_{\bar{z}}g\| \geq C_1\|g\|$ and $\|\partial_{\bar{z}}f\| \geq C_1\|f\|$. Hence, there exist $\epsilon > 0$ and $R_2 > R$ such that the following holds if $|w| > R_2$:

$$-\partial_w\partial_{\bar{w}}(\|f\|^2 + \|g\|^2) \leq -\epsilon(\|f\|^2 + \|g\|^2) \quad (43)$$

In general, if φ is a positive L^1 -subharmonic function on Y_{R_2} , Then $\varphi(w) = O(|w|^{-2})$. Indeed, by the mean value property, we have

$$\varphi(w) \leq \frac{4}{\pi(|w| - R_2)^2} \int_{|w-w'| \leq (|w|-R_2)/2} \varphi(w') \leq \frac{C_2}{(|w| - R_2)^2}.$$

Hence, we have $\|f\|^2 + \|g\|^2 = O(|w|^{-2})$. Then, by a standard argument with (43), we obtain $\|f\|^2 + \|g\|^2 = O(\exp(-C_3|w|))$. (See the proof of Lemma 3.15.) By a bootstrapping argument, we obtain $|f(z, w)| = O(\exp(-C_4|w|))$ and $|g(z, w)| = O(\exp(-C_4|w|))$. \blacksquare

4.3.3 Finiteness

We continue to use the notation in §4.3.2. Let ω be a C^∞ -section of $E \otimes \Omega^{0,1}$ on X_R . Suppose that the support of ω is contained in $T \times \{|w| \geq R+1\}$. We set $\mathcal{D} := \bar{\partial}_E + \bar{\partial}_E^*$. Let dvol denote the volume form induced by the Euclidean metric.

Lemma 4.14 *Assume that ω and $\Delta_E\omega$ are L^2 . Then, $\bar{\partial}_E^*\omega$ and $\bar{\partial}_E\omega$ are L^2 , and we have*

$$\int h(\omega, \Delta_E\omega) \text{dvol} = \int |\mathcal{D}\omega|_h^2 \text{dvol}$$

Proof Let $g := dz d\bar{z} + dw d\bar{w}$. Let $|\cdot|_{h,g}$ denote the norm of sections of $E \otimes \Omega^\bullet$ induced by h and g . Let $\chi(t)$ be a non-negative valued C^∞ -function such that $\chi(t) = 1$ ($t \leq 0$) and $\chi(t) = 0$ ($t \geq 1$), and that $\partial_t(\chi)/\chi^{1/2}$ is also C^∞ . For a large N , we put $\chi_N(w) := \chi(\log|w| - N)$. There exists $C_1 > 0$ such that $|\partial_w\chi_N| \leq C_1|w|^{-1}$, $|\partial_{\bar{w}}\chi_N| \leq C_1|w|^{-1}$, and $|\partial_w\partial_{\bar{w}}\chi_N| \leq C_1|w|^{-2}$. We have

$$\begin{aligned} \left| \int \chi_N h(\omega, \Delta_E\omega) \text{dvol} - \int \chi_N |\mathcal{D}\omega|_h^2 \text{dvol} \right| &\leq \left(\int |\bar{\partial}\chi_N|_g^2 \chi_N^{-1} |\omega|_{h,g}^2 \text{dvol} \right)^{1/2} \left(\int \chi_N |\bar{\partial}\omega|_{h,g}^2 \text{dvol} \right)^{1/2} \\ &\quad + \left(\int |\partial\chi_N|_g^2 \chi_N^{-1} |\omega|_{h,g}^2 \text{dvol} \right)^{1/2} \left(\int \chi_N |\partial\omega|_{h,g}^2 \text{dvol} \right)^{1/2} \end{aligned} \quad (44)$$

There exist $C_i > 0$ ($i = 2, 3$) such that the following holds:

$$\int \chi_N |\mathcal{D}\omega|_{h,g}^2 \text{dvol} \leq C_2 \left(\int \chi_N |\mathcal{D}\omega|_{h,g}^2 \text{dvol} \right)^{1/2} + C_3$$

Then, the first claim of Lemma 4.14 follows. We have

$$\left| \int h(\chi_N\omega, \Delta_E\omega) \text{dvol} - \int \chi_N |\mathcal{D}\omega|_{h,g}^2 \text{dvol} \right| \leq C_4 \int |\bar{\partial}\chi_N|_g |\omega|_{h,g} |\bar{\partial}\omega|_{h,g} \text{dvol} + C_4 \int |\partial\chi_N|_g |\omega|_{h,g} |\partial\omega|_{h,g} \text{dvol}$$

for some $C_4 > 0$. By the first claim, the integrands of the right hand side are dominated by some integrable functions, independently from N . By taking the limit, we obtain the second claim. \blacksquare

4.3.4 Proof of Proposition 4.12

Let us return to the setting in §4.3.1. According to Lemma 4.13, we have only to show the following lemma to establish Proposition 4.12.

Lemma 4.15 $\bar{\partial}_E\omega = \bar{\partial}_E^*\omega = 0$.

Proof By the first claim of Lemma 4.14, we obtain that $\mathcal{D}\omega$ is L^2 . By the argument in the proof of the second claim of the same lemma, we obtain $\int |\mathcal{D}\omega|_{h,g}^2 \text{dvol} = 0$, i.e., $\mathcal{D}\omega = 0$. \blacksquare

4.4 Nahm transform for L^2 -instantons

Let (E, ∇, h) be an L^2 -instanton on $T \times \mathbb{C}$. Let $D := \mathcal{S}p_\infty(E)$. We shall construct a harmonic bundle on $T^\vee \setminus D$. For any $\zeta \in T^\vee \setminus D$, let $\mathcal{L}_{-\zeta} = (\mathbb{C}, \bar{\partial}_T - \zeta d\bar{z})$ denote the corresponding line bundle on T with the natural hermitian metric. Let $\text{Nahm}(E, \nabla)_\zeta$ denote the space of L^2 -harmonic 1-forms of $E \otimes \mathcal{L}_{-\zeta}$. It is finite dimensional, and naturally isomorphic to $H^1(T \times \mathbb{P}^1, \mathcal{P}_{-1}E \otimes \mathcal{L}_{-\zeta}) \simeq H^1(T \times \mathbb{P}^1, \mathcal{P}_0E \otimes \mathcal{L}_{-\zeta})$. (See §4.2.) The Euclidean metric $dz d\bar{z} + dw d\bar{w}$ of $T \times \mathbb{C}$ and the hermitian metric h of E induce a metric $h_1(\zeta)$ of $\text{Nahm}(E, \nabla)_\zeta$. The multiplication of $-w \in \mathcal{O}_{\mathbb{P}^1}(1)$ induces an endomorphism $F_w(\zeta)$ of $\text{Nahm}(E, \nabla)_\zeta$. It is also described as $-P_\zeta \circ w$, where P_ζ denotes the orthogonal projection of the space of L^2 -sections of $E \otimes \mathcal{L}_{-\zeta} \otimes \Omega_{T \times \mathbb{C}}^{0,1}$ onto $\text{Nahm}(E, \nabla)_\zeta$. (Note Proposition 4.12.)

Let $A^{p,q}(E \otimes \mathcal{L}_{-\zeta})$ denote the space of L^2 -sections of $E \otimes \mathcal{L}_{-\zeta} \otimes \Omega_{T \times \mathbb{C}}^{p,q}$. Let $\bar{\partial}_{E,\zeta}$ denote the $\bar{\partial}$ -operator of $E \otimes \mathcal{L}_{-\zeta}$. Let $\bar{\partial}_{E,\zeta}^*$ denote its adjoint. Let $\mathcal{D}_\zeta := \bar{\partial}_{E,\zeta} + \bar{\partial}_{E,\zeta}^*$ be the closed operator $A^{0,0}(E \otimes \mathcal{L}_{-\zeta}) \oplus A^{0,2}(E \otimes \mathcal{L}_{-\zeta}) \rightarrow A^{0,1}(E \otimes \mathcal{L}_{-\zeta})$, and let $\mathcal{D}_\zeta := \bar{\partial}_{E,\zeta} + \bar{\partial}_{E,\zeta}^*$ denote its adjoint $A^{0,1}(E \otimes \mathcal{L}_{-\zeta}) \rightarrow A^{0,0}(E \otimes \mathcal{L}_{-\zeta}) \oplus A^{0,2}(E \otimes \mathcal{L}_{-\zeta})$. By the results in §4.2, we obtain that \mathcal{D}^* is surjective. We have $\text{Ker}(\mathcal{D}_\zeta) = \text{Nahm}(E, \nabla)_\zeta$. We obtain that the family $\bigcup_\zeta \text{Nahm}(E, \nabla)_\zeta$ gives a C^∞ -bundle $\text{Nahm}(E, \nabla)$ on $T^\vee \setminus D$. It is equipped with a C^∞ -metric h_1 and a C^∞ -endomorphism F_w . It is also equipped with the induced unitary connection ∇_1 . The C^∞ -bundle $\text{Nahm}(E, \nabla)$ is also constructed as a family of the cohomology of the complexes of the closed operators $(A^{0,\bullet}(E \otimes \mathcal{L}_{-\zeta}), \bar{\partial}_{E,\zeta})$. It induces a holomorphic structure of $\text{Nahm}(E, \nabla)$ as a bundle on $T^\vee \setminus D$, and F_w is holomorphic. We set $\theta_1 := F_w d\bar{\zeta}$. The $(0,1)$ -part of ∇_1 is equal to the $\bar{\partial}$ -operator of $\text{Nahm}(E, \nabla)$.

Proposition 4.16 $(E_1, \bar{\partial}_{E_1}, \theta_1, h_1)$ is a wild harmonic bundle.

Proof Because the argument is rather standard, we give only an indication for the convenience of the readers. For $I \subset \{1, 2, 3\}$, let p_I denote the projection of $T^\vee \times T \times \mathbb{P}^1$ onto the product of the i -th components. By the construction, we have a natural isomorphism $(E_1, \bar{\partial}_{E_1}) \simeq Rp_{1*}(p_{23}^* \mathcal{P}_0 E \otimes p_{12}^* \mathcal{P}oin^{-1})|_{T^\vee \setminus D}$. The endomorphism F_w is equal the multiplication of $-w : Rp_{1*}(p_{23}^* \mathcal{P}_{-1} E \otimes p_{12}^* \mathcal{P}oin^{-1})|_{T^\vee \setminus D} \rightarrow Rp_{1*}(p_{23}^* \mathcal{P}_0 E \otimes p_{12}^* \mathcal{P}oin^{-1})|_{T^\vee \setminus D}$. Hence, we obtain that θ is a wild Higgs field in the sense that, for the local expression $\theta = f d\zeta$ around $P \in D$, the coefficients of $\det(t \text{id} - f)$ are meromorphic at P .

Let us prove that $(E_1, \bar{\partial}_{E_1}, \theta_1, h_1)$ is a harmonic bundle. Although we follow a standard argument, we give rather details for the convenience of readers. Let Δ_E denote the Laplacian on $A^{0,0}(E)$, i.e., $\Delta_E = \bar{\partial}_E^* \bar{\partial}_E = -\sqrt{-1} \Lambda \partial_E \bar{\partial}_E$. We have

$$\Delta_E \psi = -2(\nabla_z \nabla_{\bar{z}} + \nabla_w \nabla_{\bar{w}}) \psi.$$

On $A^{0,2}(E)$, the Laplacian is given by $\bar{\partial}_E \bar{\partial}_E^* = (-\sqrt{-1}) \bar{\partial}_E \Lambda \partial_E$. We have

$$\bar{\partial}_E \bar{\partial}_E^* (\psi d\bar{z} d\bar{w}) = -2(\nabla_{\bar{z}} \nabla_z + \nabla_{\bar{w}} \nabla_w) \psi d\bar{z} d\bar{w}.$$

Because $F_{z\bar{z}} + F_{w\bar{w}} = 0$, it is equal to $\Delta_E(\psi) d\bar{z} d\bar{w}$. Hence, under the natural identification $A^{0,0}(E) \oplus A^{0,2}(E) \simeq A^{0,0}(E) \otimes \langle\langle 1, d\bar{z} d\bar{w} \rangle\rangle$, the Laplacian $\mathcal{D}^* \mathcal{D}$ acts as $\Delta_E \otimes \text{id}$, where $\langle\langle a, b \rangle\rangle$ denotes the 2-dimensional vector space generated by a, b . The Green operator of $\mathcal{D}^* \mathcal{D}$ acts as $\mathcal{G}_E \otimes \text{id}$, where \mathcal{G}_E denotes the Green operator for Δ_E on $A^{0,0}(E)$.

We naturally identify $A^{p,q}(E \otimes \mathcal{L}_{-\zeta})$ with $A^{p,q}(E)$. For a differential form τ , let $\mu(\tau)$ be an endomorphism of $\bigoplus A^{p,q}(E)$ given by $\mu(\tau)(\varphi) = \tau \wedge \varphi$. We have $\bar{\partial}_{E,\zeta} = \bar{\partial}_E - \zeta \mu(d\bar{z})$ and $\bar{\partial}_{E,\zeta}^* = \bar{\partial}_E^* + (\sqrt{-1}) \bar{\zeta} \Lambda \circ \mu(dz)$. Let d_{T^\vee} denote the trivial connection of the product vector bundle $A^{0,1}(E) \times (T^\vee \setminus D)$ over $T^\vee \setminus D$. We have the following relation for the operators on the space of the sections $T^\vee \setminus D \rightarrow A^{0,1}(E) \times (T^\vee \setminus D)$:

$$[d_{T^\vee}, \bar{\partial} + \zeta d\bar{z}] = d\zeta \mu(d\bar{z}), \quad [d_{T^\vee}, (\bar{\partial} + \zeta d\bar{z})^*] = \sqrt{-1} d\bar{\zeta} \Lambda \circ \mu(dz).$$

We set $\Omega := d\zeta \mu(d\bar{z}) + d\bar{\zeta} \sqrt{-1} \Lambda \circ \mu(dz)$. Let P_ζ denote the orthogonal projection of $A^{0,1}(E)$ onto the kernel of \mathcal{D}_ζ^* . Let $\Delta_\zeta = \bar{\partial}_{E,\zeta}^* \bar{\partial}_{E,\zeta}$ denote the Laplacian on $A^{0,0}(E)$ for $E \otimes \mathcal{L}_{-\zeta}$. Let \mathcal{G}_ζ denote the Laplacian for Δ_ζ on $A^{0,0}(E)$, i.e., $\mathcal{G}_\zeta \Delta_\zeta = \text{id}_{A^{0,0}(E)}$. The Green operator G_ζ for $\mathcal{D}_\zeta^* \mathcal{D}_\zeta$ on $A^{0,0} \oplus A^{0,2}$ is given by $\mathcal{G}_\zeta \otimes \text{id}$. We have $P_\zeta = 1 - \mathcal{D}_\zeta \circ G_\zeta \circ \mathcal{D}_\zeta^*$. Let $\mathcal{G}_\zeta \otimes \text{id}$ also denote the naturally induced operator on $A^{0,1} \simeq A^{0,0} \otimes \langle\langle d\bar{z}, d\bar{w} \rangle\rangle$.

Let $\langle \cdot, \cdot \rangle$ denote the inner product of $A^{0,\bullet}(E)$ induced by h and $dz d\bar{z} + dw d\bar{w}$. By a standard computation, the curvature F of the connection ∇_1 is described as follows, for any sections ψ_i ($i = 1, 2$) of $\text{Nahm}(E, \nabla)$:

$$\begin{aligned} \langle \psi_1, F\psi_2 \rangle &= \langle \psi_1, d_{T^\vee} \circ P_\zeta(d_{T^\vee} \psi_2) \rangle = -\langle \psi_1, d_{T^\vee} \circ \mathcal{D}_\zeta \circ G_\zeta \circ \mathcal{D}_\zeta^*(d_{T^\vee} \psi_2) \rangle \\ &= \langle d_{T^\vee} \psi_1, \mathcal{D}_\zeta \circ G_\zeta \circ \mathcal{D}_\zeta^*(d_{T^\vee} \psi_2) \rangle = \langle \mathcal{D}_\zeta^* d_{T^\vee} \psi_1, G_\zeta \circ \mathcal{D}_\zeta^*(d_{T^\vee} \psi_2) \rangle = \langle \Omega \psi_1, G_\zeta \Omega \psi_2 \rangle \\ &= d\zeta d\bar{\zeta} \left(\langle d\bar{z} \psi_1, d\bar{z} (\mathcal{G}_\zeta \otimes \text{id}) \psi_2 \rangle - \langle \Lambda(dz \psi_1), \Lambda(dz (\mathcal{G}_\zeta \otimes \text{id}) \psi_2) \rangle \right) \end{aligned} \quad (45)$$

We have $\theta(\psi) = P_\zeta(w\psi)d\zeta$ and $\theta^\dagger(\psi) = P_\zeta(\bar{w}\psi)d\bar{\zeta}$. We have

$$\begin{aligned} \langle \psi_1, (P_\zeta w \circ P_\zeta \bar{w} - P_\zeta \bar{w} \circ P_\zeta w) \psi_2 \rangle d\bar{\zeta} d\zeta &= -\langle \psi_1, (w(P_\zeta - 1)\bar{w} - \bar{w}(P_\zeta - 1)w) \psi_2 \rangle d\zeta d\bar{\zeta} \\ &= \left(\langle \bar{w} \psi_1, \mathcal{D}_\zeta G_\zeta \mathcal{D}_\zeta^* \bar{w} \psi_2 \rangle - \langle w \psi_1, \mathcal{D}_\zeta G_\zeta \mathcal{D}_\zeta^* w \psi_2 \rangle \right) d\zeta d\bar{\zeta} \\ &= \left(\langle \mathcal{D}_\zeta^*(\bar{w} \psi_1), G_\zeta \mathcal{D}_\zeta^*(\bar{w} \psi_2) \rangle - \langle \mathcal{D}_\zeta^*(w \psi_1), G_\zeta \mathcal{D}_\zeta^*(w \psi_2) \rangle \right) d\zeta d\bar{\zeta} \\ &= \left(\langle [\mathcal{D}_\zeta^*, \bar{w}] \psi_1, G_\zeta [\mathcal{D}_\zeta^*, \bar{w}] \psi_2 \rangle - \langle [\mathcal{D}_\zeta^*, w] \psi_1, G_\zeta [\mathcal{D}_\zeta^*, w] \psi_2 \rangle \right) d\zeta d\bar{\zeta} \end{aligned} \quad (46)$$

We have $[\mathcal{D}_\zeta^*, \bar{w}] = \mu(d\bar{w})$ and $[\mathcal{D}_\zeta^*, w] = -\sqrt{-1}\Lambda \circ \mu(dw)$. Hence, we obtain the following expression:

$$\langle \psi_1, (P_\zeta w \circ P_\zeta \bar{w} - P_\zeta \bar{w} \circ P_\zeta w) \psi_2 d\zeta d\bar{\zeta} \rangle = \left(\langle d\bar{w} \psi_1, d\bar{w} (\mathcal{G}_\zeta \otimes \text{id}) \psi_2 \rangle - \langle \Lambda(dw \psi_1), \Lambda(dw (\mathcal{G}_\zeta \otimes \text{id}) \psi_2) \rangle \right) d\zeta d\bar{\zeta} \quad (47)$$

By using $\langle d\bar{z} d\bar{w}, d\bar{z} d\bar{w} \rangle = \langle \Lambda dw d\bar{w}, \Lambda dw d\bar{w} \rangle = \langle \Lambda dz d\bar{z}, \Lambda dz d\bar{z} \rangle$ for the metric on $\Omega_{T \times \mathbb{C}}^\bullet$, we obtain the following:

$$\langle \psi_1, (F + (P \circ w d\zeta) \circ (P \circ \bar{w} d\bar{\zeta})) \psi_2 \rangle = 0$$

Namely, the Hitchin equation is satisfied. Thus, the proof of Proposition 4.16 is finished. \blacksquare

Remark 4.17 We obtain a different transformation by replacing $\mathcal{L}_{-\zeta}$ with \mathcal{L}_ζ , for which we do not need any essential change.

Remark 4.18 We use the operators, which looks natural in the complex geometry, instead of the Dirac operator itself.

5 Algebraic Nahm transform

5.1 Local algebraic Nahm transform

5.1.1 Admissible filtered Higgs bundles

Let $X := \{z \in \mathbb{C} \mid |z| < \rho_0\}$ and $D := \{0\}$. For each positive integer p , let $\varphi_p : X^{(p)} = \{|z_p| < \rho_0^{1/p}\} \rightarrow X$ be given by $\varphi_p(z_p) = z_p^p$. Let $\mathcal{P}_* \mathcal{E}$ be a filtered bundle on (X, D) with a Higgs field θ . Let $m \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{Z}_{>0}$ such that $\text{g.c.d.}(p, m) = 1$. We say that $(\mathcal{P}_* V, \theta)$ has slope (p, m) , if the following holds:

- Let $(\mathcal{P}_* V^{(p)}, \theta^{(p)})$ be a filtered Higgs bundle obtained as the pull back of $(\mathcal{P}_* V, \theta)$ by φ_p . Then, we have $z_p^m \theta^{(p)}(\mathcal{P}_c V^{(p)}) \subset \mathcal{P}_c V^{(p)} dz_p / z_p$ for any $c \in \mathbb{R}$.
- Let $\text{Res}(z_p^m \theta^{(p)})$ denote the endomorphism of $\text{Gr}_c^{\mathcal{P}}(V^{(p)})$ obtained as the residue of $z_p^m \theta^{(p)}$. If $(p, m) \neq (1, 0)$, we impose that $\text{Res}(z_p^m \theta^{(p)})$ is invertible for any c .

Although $\text{Res}(z_p^m \theta^{(p)})$ may depend on the choice of a coordinate, but the above condition is independent. Let $\mathcal{I}(\theta)$ denote the set of the eigenvalues of $\text{Res}(z_p^m \theta^{(p)})$. We have $\text{Gal}(\varphi_p)$ -action on $(\mathcal{P}_* V^{(p)}, \theta^{(p)})$ and $\mathcal{I}(\theta)$. The quotient set $\mathcal{I}(\theta) / \text{Gal}(\varphi)$ is denoted by $\mathcal{I}(\theta)$. We have the orbit decomposition $\mathcal{I}(\theta) = \coprod_{\mathbf{o} \in \mathcal{I}(\theta)} \mathbf{o}$. We say that $(\mathcal{P}_* V, \theta)$ has type (p, m, \mathbf{o}) , if moreover $\mathcal{I}(\theta) = \{\mathbf{o}\}$.

If $m \neq 0$, then \mathbf{o} is an element of $\mathcal{J}(p, m) := \mathbb{C}^*/\text{Gal}(\varphi_p)$, where the action is given by $(t, \alpha) \mapsto t^m \alpha$. If $m = 0$, then \mathbf{o} is an element of $\mathcal{J}(1, 0) := \mathbb{C}$. When (\mathcal{P}_*V, θ) has slope (p, m) , it has a decomposition $(\mathcal{P}_*V, \theta) = \bigoplus_{\mathbf{o} \in \mathcal{J}(p, m)} (\mathcal{P}_*V_{\mathbf{o}}, \theta_{\mathbf{o}})$ after X is shrunk appropriately, such that $(\mathcal{P}_*V_{\mathbf{o}}, \theta_{\mathbf{o}})$ has type (p, m, \mathbf{o}) .

We say that (\mathcal{P}_*V, θ) is admissible, if it has a decomposition $(\mathcal{P}_*V, \theta) = \bigoplus_{(p, m)} (\mathcal{P}_*V^{(p, m)}, \theta^{(p, m)})$ after X is shrunk appropriately, such that $(\mathcal{P}_*V^{(p, m)}, \theta^{(p, m)})$ has slope (p, m) . The decomposition is called the slope decomposition. We may also have a refined decomposition $(\mathcal{P}_*V, \theta) = \bigoplus_{(p, m, \mathbf{o})} (\mathcal{P}_*V_{\mathbf{o}}^{(p, m)}, \theta_{\mathbf{o}}^{(p, m)})$ such that $(\mathcal{P}_*V_{\mathbf{o}}^{(p, m)}, \theta_{\mathbf{o}}^{(p, m)})$ has type (p, m, \mathbf{o}) . It is called the type decomposition. For $\alpha \in \mathbb{Q}_{\geq 0}$, we say that the slope of (\mathcal{P}_*V, θ) is smaller (resp. strictly smaller) than α , if $\mathcal{P}_*V^{(p, m)} = 0$ for $p/m > \alpha$ (resp. $p/m \geq \alpha$) in the slope decomposition.

Suppose that (\mathcal{P}_*V, θ) has type (p, m, \mathbf{o}) . After X is shrunk appropriately, we have the decomposition $(\mathcal{P}_*V^{(p)}, \theta^{(p)}) = \bigoplus_{\alpha \in \mathbf{o}} (\mathcal{P}_*V_{\alpha}^{(p)}, \theta_{\alpha}^{(p)})$ such that $\text{Res}(z_p^m \theta_{\alpha}^{(p)})$ has a unique eigenvalue α . For any $\alpha \in \mathbf{o}$, we have a natural isomorphism $\varphi_{p*}(\mathcal{P}_*V_{\alpha}^{(p)}, \theta_{\alpha}^{(p)}) \simeq (\mathcal{P}_*V, \theta)$.

In the following, we shall shrink X without mentioning it.

5.1.2 Complex

We define a complex of sheaves associated to an admissible filtered Higgs bundle (\mathcal{P}_*V, θ) . First, let us consider the case that (\mathcal{P}_*V, θ) has type (p, m, \mathbf{o}) . Suppose $(p, m, \mathbf{o}) \neq (1, 0, 0)$. For each $c \in \mathbb{R}$, let $\mathcal{P}_c(V \otimes \Omega_X^{\bullet}, \theta)$ denote the complex

$$\mathcal{P}_{c-m/p}V \longrightarrow \mathcal{P}_{c+1}V dz$$

where the first term sits in the degree 0. Take any $\alpha \in \mathbf{o}$. For each $c \in \mathbb{R}$, let $\mathcal{P}_c(V_{\alpha}^{(p)} \otimes \Omega_{X^{(p)}}^{\bullet}, \theta_{\alpha}^{(p)})$ denote the following complex on $X^{(p)}$:

$$\mathcal{P}_{c-m}V_{\alpha}^{(p)} \xrightarrow{\theta_{\alpha}^{(p)}} \mathcal{P}_cV_{\alpha}^{(p)} \otimes \frac{dz_p}{z_p}.$$

We have a natural isomorphism $\mathcal{P}_c(V \otimes \Omega^{\bullet}, \theta) \simeq \varphi_{p*}\mathcal{P}_{cp}(V_{\alpha}^{(p)} \otimes \Omega_{X^{(p)}}^1, \theta_{\alpha}^{(p)})$. It is also isomorphic to the descent of $\bigoplus_{\alpha \in \mathbf{o}} \mathcal{P}_{cp}(V_{\alpha}^{(p)} \otimes \Omega_{X^{(p)}}^1, \theta_{\alpha}^{(p)})$. For $c \leq c'$, the natural inclusion $\mathcal{P}_c(V \otimes \Omega^{\bullet}, \theta) \longrightarrow \mathcal{P}_{c'}(V \otimes \Omega^{\bullet}, \theta)$ is a quasi-isomorphism. We set $\mathcal{C}^{\bullet}(\mathcal{P}_*V, \theta) := \mathcal{P}_{-1/2}(V \otimes \Omega^{\bullet}, \theta)$.

In the case $(p, m, \mathbf{o}) = (1, 0, 0)$, we set $\mathcal{C}^0(\mathcal{P}_*V, \theta) := \mathcal{P}_0V$ and

$$\mathcal{C}^1(\mathcal{P}_*V, \theta) := \mathcal{P}_{<1}V \otimes \Omega_X^1 + \theta(\mathcal{P}_0V) \subset \mathcal{P}_1V \otimes \Omega_X^1.$$

Thus, we obtain the complex $\mathcal{C}^{\bullet}(\mathcal{P}_*V, \theta)$, when (\mathcal{P}_*V, θ) has type (p, m, \mathbf{o}) .

For a general admissible filtered Higgs bundle (\mathcal{P}_*V, θ) , the complex $\mathcal{C}^{\bullet}(\mathcal{P}_*V, \theta)$ is defined as the extension of the complex $(V \longrightarrow V \otimes \Omega_X^1)$ on $X \setminus D$ to a complex on X , such that it is $\bigoplus_{(m, p, \mathbf{o})} \mathcal{C}^{\bullet}(\mathcal{P}_*V_{\mathbf{o}}^{(p, m)}, \theta_{\mathbf{o}}^{(p, m)})$ around D , according to the type decomposition.

Lemma 5.1 *If (\mathcal{P}_*V, θ) comes from a wild harmonic bundle $(E, \bar{\partial}_E, \theta, h)$ on (X, D) , then $\bigoplus \mathcal{C}^{\bullet}(\mathcal{P}_*V, \theta)$ is naturally quasi-isomorphic to the complex of square-integrable sections of the Higgs complex $E \otimes \Omega^{\bullet}$.*

Proof By the descent, we have only to consider the unramified case. We omit to denote p . We have the naturally defined map $\pi_c : \mathcal{P}_cV \longrightarrow \text{Gr}_c^{\mathcal{P}}(V)$. Let W be the weight filtration of the nilpotent part of the endomorphism $\text{Res}(\theta)$ on $\text{Gr}_c^{\mathcal{P}}(V)$. We set $W_k \mathcal{P}_cV := \pi_c^{-1}(W_k \text{Gr}_c^{\mathcal{P}}(V))$.

Suppose that (\mathcal{P}_*V, θ) has type (m, \mathbf{o}) . If $(m, \mathbf{o}) \neq (0, 0)$, let $\mathcal{C}_{L^2}^{\bullet}(\mathcal{P}_*V, \theta)$ be the following complex:

$$W_{-2}\mathcal{P}_{-m}V \longrightarrow W_{-2}\mathcal{P}_0V \otimes \Omega_X^1(\log D)$$

It is easy to check that the following natural morphisms are quasi-isomorphisms:

$$\mathcal{C}_{L^2}^{\bullet}(\mathcal{P}_*V, \theta) \longrightarrow \mathcal{P}_0(V \otimes \Omega^{\bullet}, \theta) \longleftarrow \mathcal{C}^{\bullet}(\mathcal{P}_*V, \theta)$$

If $(m, \mathbf{o}) = (0, 0)$, let $\mathcal{C}_{L^2}^{\bullet}(\mathcal{P}_*V, \theta)$ be the following complex:

$$W_0\mathcal{P}_0V \longrightarrow W_{-2}\mathcal{P}_0V \otimes \Omega_X^1(\log D)$$

It is easy to check that the natural inclusion $\mathcal{C}_{L^2}^\bullet(\mathcal{P}_*V, \theta) \longrightarrow \mathcal{C}^\bullet(\mathcal{P}_*V, \theta)$ is a quasi-isomorphism.

In general, we define $\mathcal{C}_{L^2}^\bullet(\mathcal{P}_*V, \theta) = \bigoplus \mathcal{C}_{L^2}^\bullet(\mathcal{P}_*V_{\mathbf{o}}^{(m)}, \theta_{\mathbf{o}}^{(m)})$ by using the type decomposition. According to the result in §5.1 of [38], $\mathcal{C}_{L^2}^\bullet(\mathcal{P}_*V, \theta)$ is naturally quasi-isomorphic to the complex of square-integrable sections of the Higgs complex $E \otimes \Omega^\bullet$. Thus, we obtain the claim of the lemma. \blacksquare

5.1.3 Transform

We shall construct some transformations for filtered Higgs bundles, which are analogue to the local Fourier transform in [14] and [21].

In the following, for a variable x , let U_x denote a small neighbourhood of 0 in \mathbb{C}_x . For two variables x and y , let $U_{x,y} := U_x \times U_y$, and let $\pi_1 : U_{x,y} \longrightarrow U_x$ and $\pi_2 : U_{x,y} \longrightarrow U_y$ denote the projections.

Let (\mathcal{P}_*V, θ) be an admissible filtered Higgs bundle on $(U_\zeta, 0)$. Let us define a filtered bundle $\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta)$ with an endomorphism g on U_τ . We consider the following complex on $U_{\zeta,\tau}$:

$$\pi_1^* \mathcal{C}^0(\mathcal{P}_*V, \theta) \xrightarrow{\tau\theta + d\zeta} \pi_1^* \mathcal{C}^1(\mathcal{P}_*V, \theta)$$

Let \mathcal{Q} be the quotient. If U_τ is sufficiently small, then the support of \mathcal{Q} is proper over U_τ . We define

$$\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta) := \pi_{2*} \mathcal{Q}, \quad \mathcal{N}^{0,\infty}(\mathcal{P}_*V, \theta) := \mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta)(*\tau).$$

Here, $(*\tau)$ means the localization with respect to τ . It is easy to check that $\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta)$ is coherent and torsion-free. Hence, $\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta)$ is a locally free \mathcal{O}_{U_τ} -module. In particular, $\mathcal{N}^{0,\infty}(\mathcal{P}_*V, \theta)$ is a locally free $\mathcal{O}_{U_\tau}(*\tau)$ -module. The multiplication of ζ induces the endomorphism g . By setting $\psi := -g\tau^{-2}d\tau$, we obtain a Higgs field of $\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta)$. We shall introduce a filtered bundle $\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta) = (\mathcal{N}_a^{0,\infty}(\mathcal{P}_*V, \theta) \mid a \in \mathbb{R})$ over $\mathcal{N}^{0,\infty}(\mathcal{P}_*V, \theta)$.

If (\mathcal{P}_*V, θ) has type $(p, m, \mathbf{o}) \neq (1, 0, 0)$, we consider the following complexes on $U_{\zeta,\tau}$ for any $c \in \mathbb{R}$:

$$\pi_1^* \mathcal{P}_{c-m/p}(V) \xrightarrow{\tau\theta + d\zeta} \pi_1^* \mathcal{P}_c(V)(d\zeta/\zeta) \quad (48)$$

Let \mathcal{Q}_c denote the quotient. We define

$$\mathcal{N}_{\kappa_1(p,m,c)}^{0,\infty}(\mathcal{P}_*V, \theta) := \pi_{2*} \mathcal{Q}_c, \quad \kappa_1(p, m, c) := \frac{2pc - m}{2(p + m)}.$$

By construction, we have $\mathcal{N}_{-1/2}^{0,\infty}(\mathcal{P}_*V, \theta) = \mathcal{N}(\mathcal{P}_*V, \theta)$ in this case. It is easy to check that $\mathcal{N}_a^{0,\infty}(\mathcal{P}_*V, \theta)$ are locally free \mathcal{O}_{U_τ} -module of finite rank. We have a naturally induced map $\mathcal{N}_{a'}^{0,\infty}(\mathcal{P}_*V, \theta) \longrightarrow \mathcal{N}_a^{0,\infty}(\mathcal{P}_*V, \theta)$ for $a' \leq a$. Its restriction to $\{\tau \neq 0\}$ is an isomorphism, and hence it is injective. We also obtain $\mathcal{N}_a^{0,\infty}(\mathcal{P}_*V, \theta)(*\tau) = \mathcal{N}_0^{0,\infty}(\mathcal{P}_*V, \theta)$. For $c' := c - (1 + p/m)$, the images of $\tau \cdot \pi_1^* \mathcal{P}_c(V)(d\zeta/\zeta)$ and $\pi_1^* \mathcal{P}_{c'}(V)(d\zeta/\zeta)$ are the same in the quotient of \mathcal{Q}_c . It implies $\tau \mathcal{N}_a^{0,\infty}(\mathcal{P}_*V, \theta) = \mathcal{N}_{a-1}^{0,\infty}(\mathcal{P}_*V, \theta)$ for any $a \in \mathbb{R}$. Hence, $\mathcal{N}_a^{0,\infty}(\mathcal{P}_*V, \theta)$ ($a \in \mathbb{R}$) give a filtered bundle over $\mathcal{N}^{0,\infty}(\mathcal{P}_*V, \theta)$.

If (\mathcal{P}_*V, θ) has type $(p, m, \mathbf{o}) = (1, 0, 0)$, we define $\mathcal{N}_0^{0,\infty}(\mathcal{P}_*V, \theta) := \mathcal{N}^{0,\infty}(\mathcal{P}_*V, \theta)$. We have the following natural morphisms:

$$\mathcal{N}_0^{0,\infty}(\mathcal{P}_*V, \theta)|_0 \simeq \mathcal{C}^1(\mathcal{P}_*V, \theta)/\mathcal{C}^0(\mathcal{P}_*V, \theta)d\zeta \longrightarrow (\mathcal{P}_0V)|_0$$

Here, the subscript “|0” means the fiber of the vector bundle over 0, and the latter map is given by the residue, which is injective. Hence, the parabolic filtration of the right hand side induces a parabolic filtration of $\mathcal{N}_0^{0,\infty}(\mathcal{P}_*V, \theta)|_0$ indexed by $] - 1, 0]$. It induces a filtered bundle $\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta)$ over $\mathcal{N}^{0,\infty}(\mathcal{P}_*V, \theta)$.

If (\mathcal{P}_*V, θ) is admissible, we replace U_ζ with smaller neighbourhoods so that it has the type decomposition, and we define

$$\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta) := \bigoplus_{p,m,\mathbf{o}} \mathcal{N}_*^{0,\infty}(\mathcal{P}_*V_{\mathbf{o}}^{(p,m)}, \theta).$$

The construction $\mathcal{N}_*^{0,\infty}$ gives a functor from the category of the germs of admissible filtered Higgs bundles to the category of the germs of filtered Higgs bundles. We set $\mathcal{N}_{<a}^{0,\infty}(\mathcal{P}_*V, \theta) := \sum_{b < a} \mathcal{N}_b^{0,\infty}(\mathcal{P}_*V, \theta)$.

Lemma 5.2 Suppose (\mathcal{P}_*V, θ) has type (p, m, \mathbf{o}) . The rank of $\mathcal{N}^{0,\infty}(\mathcal{P}_*V, \theta)$ is $(p+m)\text{rank } V/p$ in the case $(p, m, \mathbf{o}) \neq (1, 0, 0)$, or $\text{rank } V - \dim \text{Ker } \text{Gr}_0^{\mathcal{P}}(\text{Res } \theta)$ in the case $(p, m, \mathbf{o}) = (1, 0, 0)$.

Proof The rank is equal to the dimension of $\mathcal{C}^1(\mathcal{P}_*V, \theta)/\mathcal{C}^0(\mathcal{P}_*V, \theta)d\zeta$ as a \mathbb{C} -vector space. Then, the claim can be checked by a direct computation. (See also the proof of Lemma 5.3 below for the case $(p, m, \mathbf{o}) \neq (1, 0, 0)$.) ■

Lemma 5.3 $(\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta), \psi)$ is admissible. If (\mathcal{P}_*V, θ) has type (p, m, \mathbf{o}) , then $(\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta), \psi)$ has type $(p+m, m, \mathbf{o}')$ for some \mathbf{o}' .

Proof We have only to consider the case that (\mathcal{P}_*V, θ) has type (p, m, \mathbf{o}) . Let us consider the case $(p, m, \mathbf{o}) = (1, 0, 0)$. For the expression $\theta = f d\zeta/\zeta$, f gives an endomorphism of \mathcal{P}_cV for any c , and $f|_0$ is nilpotent. Because $\psi = -\tau^{-1}g(d\tau/\tau) = f(d\tau/\tau)$, $-\tau^{-1}g$ is induced by f , it preserves $\mathcal{N}_0^{0,\infty}(\mathcal{P}_*V, \theta)$. If we regard $\mathcal{N}_0^{0,\infty}(\mathcal{P}_*V, \theta)|_0$ as a subspace of $\mathcal{P}_0V|_0$ as above, $(-\tau^{-1}g)|_0$ is the restriction of $f|_0$. Hence, it is nilpotent, i.e., $(\mathcal{N}^{0,\infty}(\mathcal{P}_*V, \theta), \psi)$ has type $(1, 0, 0)$.

Let us consider the case $(p, m, \mathbf{o}) \neq (1, 0, 0)$. Fix $\alpha \in \mathbf{o}$. We consider the following on $U_{\zeta_p, \tau}$:

$$\pi_1^* \mathcal{P}_{pc-m} V_\alpha^{(p)} \xrightarrow{\tau \theta_\alpha^{(p)} + d\zeta_p^p} \pi_1^* \mathcal{P}_{pc} V_\alpha^{(p)} (d\zeta_p/\zeta_p) \quad (49)$$

The quotient is denoted by \mathcal{Q}'_c . We have a natural isomorphism $\pi_{2*} \mathcal{Q}'_c \simeq \mathcal{N}_{\kappa_1(p, m, c)}^{0,\infty}(\mathcal{P}_*V, \theta)$.

The natural map $\mathcal{Q}'_{c'} \rightarrow \mathcal{Q}'_c$ ($c' \leq c$) is injective. We set $\mathcal{Q}'_{<c} := \bigcup_{b < c} \mathcal{Q}'_b$. We have the following exact sequence:

$$0 \longrightarrow \pi_1^* \text{Gr}_{pc-m}^{\mathcal{P}}(V_\alpha^{(p)}) \xrightarrow{\tau \theta_\alpha^{(p)}} \pi_1^* \text{Gr}_{pc}^{\mathcal{P}}(V_\alpha^{(p)}) \longrightarrow \mathcal{Q}'_c/\mathcal{Q}'_{<c} \longrightarrow 0$$

It induces the following isomorphism of \mathbb{C} -vector spaces for any $c \in \mathbb{R}$:

$$\text{Gr}_{pc}^{\mathcal{P}}(V_\alpha^{(p)}) \simeq \frac{\mathcal{N}_{\kappa_1(p, m, c)}^{0,\infty}(\mathcal{P}_*V, \theta)}{\mathcal{N}_{<\kappa_1(p, m, c)}^{0,\infty}(\mathcal{P}_*V, \theta)} \quad (50)$$

Let $\mathbf{v} = (v_i)$ be a frame of $\mathcal{P}_{pc} V_\alpha^{(p)}$. We set $c_i := \min\{a \in \mathbb{R} \mid v_i \in \mathcal{P}_a V_\alpha^{(p)}\}$. We assume that \mathbf{v} is compatible with the parabolic structure in the sense that the induced tuple $\{[v_i] \mid c_i = d\}$ of elements in $\text{Gr}_d^{\mathcal{P}}(V_\alpha^{(p)})$ is a base for any $d \in]pc-1, pc]$. We set $\nu_{ij} := \zeta_p^i v_j$ for $(0 \leq i \leq p+m-1, 1 \leq j \leq \text{rank } V_\alpha^{(p)})$. The induced sections of $\mathcal{N}_{\kappa_1(p, m, c)}^{0,\infty}(\mathcal{P}_*V, \theta)$ are also denoted by the same symbols. Because they induce a base of $\mathcal{N}_{\kappa_1(p, m, c)}^{0,\infty}(\mathcal{P}_*V, \theta)/\tau \mathcal{N}_{\kappa_1(p, m, c)}^{0,\infty}(\mathcal{P}_*V, \theta)$, they give a frame of $\mathcal{N}_{\kappa_1(p, m, c)}^{0,\infty}(\mathcal{P}_*V, \theta)$ on a neighbourhood of 0. (In particular, the rank of $\mathcal{N}_{\kappa_1(p, m, c)}^{0,\infty}(\mathcal{P}_*V, \theta)$ is $(p+m)\text{rank } V_\alpha^{(p)} = p^{-1}(p+m)\text{rank } V$.) Moreover, by the isomorphism (50), the frame is compatible with the parabolic structure of $\mathcal{N}_{\kappa_1(p, m, c)}^{0,\infty}(\mathcal{P}_*V, \theta)$.

We take a ramified covering $\varphi : U_\eta \rightarrow U_\tau$ by $\varphi(\eta) = \eta^{p+m}$. Let $\mathcal{P}_*\mathcal{V}$ be the filtered bundle on $(U_\eta, 0)$ obtained as the pull back of $\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta)$ by φ . The tuple of the sections $\tilde{\nu}_{ij} := \eta^{-i} \varphi^* \nu_{ij}$ gives a frame $\tilde{\mathbf{v}}$ of $\mathcal{P}_{pc-m/2} \mathcal{V}$ which is compatible with the parabolic structure. By the frames \mathbf{v} and $\tilde{\mathbf{v}}$, we obtain an isomorphism of $\mathcal{P}_{pc-m/2} \mathcal{V}|_0$ to $(\mathcal{P}_{pc} V_\alpha^{(p)})|_0 \otimes \mathbb{C}^{p+m}$.

Let us show that $\psi = -\tau^{-2}g d\tau$ has type $(p+m, m, \mathbf{o}')$ for some \mathbf{o}' . Note that g is induced by the multiplication of $\zeta = \zeta_p^p$. Let g_1 be the endomorphism of $\mathcal{N}^{0,\infty}(\mathcal{P}_*V, \theta)$ induced by the multiplication of ζ_p . We have $g_1(\mathcal{N}_a^{0,\infty}(\mathcal{P}_*V, \theta)) \subset \mathcal{N}_{a-1/(p+m)}^{0,\infty}(\mathcal{P}_*V, \theta)$. Hence, we obtain $\eta^{-1}g_1$ gives an endomorphism of $\mathcal{P}_{pc-m/2} \mathcal{V}$. In particular, $\eta^{-p}g$ gives an endomorphism of $\mathcal{P}_{pc-m/2} \mathcal{V}$. Let us show that the restriction $(\eta^{-p}g)|_0$ has a unique non-zero eigenvalue modulo the action of $\text{Gal}(\varphi)$.

We have the parabolic filtration F of $(\mathcal{P}_{pc} V_\alpha)|_0$. We take a refinement \tilde{F} of F such that (i) \tilde{F} is preserved by $\text{Res}(\zeta_p^m \theta_\alpha^{(p)})$, (ii) the induced endomorphism on the associated graded space $\text{Gr}^{\tilde{F}}$ is semisimple. The filtration \tilde{F} induces a filtration \tilde{F}' of $\mathcal{P}_{pc-m/2} \mathcal{V}$. Then, $(\eta^{-p}g)|_0$ is compatible with \tilde{F}' , and the induced endomorphisms on $\text{Gr}^{\tilde{F}'}$ are represented by the following matrix with respect to an appropriate base:

$$\sum_{i=1}^m I \otimes E_{p+i, i} + \sum_{i=1}^p \alpha I \otimes E_{i, m+i}$$

Here, I is the identity matrix and E_{ij} denote the $(p+m)$ -square matrix whose (k, ℓ) -entry is 1 if $(k, \ell) = (i, j)$, and 0 otherwise. Then, the set of the eigenvalues are $e^{2\pi\sqrt{-1}j/(p+m)}\alpha^p$ ($j = 0, \dots, p+m-1$). Thus, we are done. \blacksquare

Corollary 5.4 *The construction $\mathcal{N}_*^{0,\infty}$ gives a functor from the category of the germs of admissible filtered Higgs bundles to the category of the germs of admissible filtered Higgs bundles whose slopes are strictly less than 1.* \blacksquare

5.1.4 Inverse transform

Let \mathcal{P}_*V be a filtered bundle on $(U_\tau, 0)$ with an endomorphism g . We say that (\mathcal{P}_*V, g) has type (p, m, \mathbf{o}) (slope (p, m)), if $(\mathcal{P}_*V, -\tau^{-2}gd\tau)$ has type (p, m, \mathbf{o}) (resp. slope (p, m)). The condition implies $p \geq m$. We say that (\mathcal{P}_*V, g) is admissible, if $(\mathcal{P}_*V, -\tau^{-2}gd\tau)$ is admissible. We have the type decomposition $(\mathcal{P}_*V, g) = \bigoplus (\mathcal{P}_*V_{\mathbf{o}}^{(p,m)}, g_{\mathbf{o}}^{(p,m)})$ and the slope decomposition $(\mathcal{P}_*V, g) = \bigoplus (\mathcal{P}_*V^{(p,m)}, g^{(p,m)})$. In this subsection, we impose the following vanishing:

(C0) $V_0^{(1,0)} = 0$, and $V^{(p,m)} = 0$ unless $p > m$.

If (\mathcal{P}_*V, g) has slope (p, m) , we consider the following complex on $U_{\tau, \zeta}$:

$$\pi_1^* \mathcal{P}_c V \xrightarrow{g-\zeta} \pi_1^* \mathcal{P}_c V$$

The quotient is denoted by \mathcal{M}_c . If U_ζ is sufficiently small, the support of \mathcal{M}_c is proper over U_ζ . We define

$$\mathcal{N}_{\kappa_2(p,m,c)+1}^{\infty,0}(\mathcal{P}_*V, g) := \pi_{2*} \mathcal{M}_c, \quad \kappa_2(p, m, c) := \frac{2pc + m}{2(p-m)}.$$

They are locally free \mathcal{O}_{U_ζ} -modules. For $a \leq a'$, we have a naturally defined map $\mathcal{N}_a^{\infty,0}(\mathcal{P}_*V, g) \rightarrow \mathcal{N}_{a'}^{\infty,0}(\mathcal{P}_*V, g)$ which induces $\mathcal{N}_a^{\infty,0}(\mathcal{P}_*V, g)(*\zeta) \simeq \mathcal{N}_{a'}^{\infty,0}(\mathcal{P}_*V, g)(*\zeta)$. We have $\mathcal{N}_{a-1}^{\infty,0}(\mathcal{P}_*V, g) = \zeta \mathcal{N}_a^{\infty,0}(\mathcal{P}_*V, g)$ for any $a \in \mathbb{R}$. Thus, we obtain a filtered bundle $\mathcal{N}_*^{\infty,0}(\mathcal{P}_*V, g)$ on $(U_\zeta, 0)$. In the general case, we define $\mathcal{N}_*^{\infty,0}(\mathcal{P}_*V, g) := \bigoplus \mathcal{N}_*^{\infty,0}(\mathcal{P}_*V_{\mathbf{o}}^{(p,m)}, g_{\mathbf{o}}^{(p,m)})$ by using the slope decomposition of (\mathcal{P}_*V, g) . The multiplication of $-\tau^{-1}$ gives a meromorphic endomorphism F . We put $\theta = Fd\zeta$. The construction gives a functor from the category of the germs of admissible filtered Higgs bundles satisfying **(C0)** to the category of germs of filtered Higgs bundle.

Lemma 5.5 *$(\mathcal{N}_*^{\infty,0}(\mathcal{P}_*V, g), \theta)$ is admissible. If (\mathcal{P}_*V, g) has type (p, m, \mathbf{o}) , then $\mathcal{N}_*^{\infty,0}(\mathcal{P}_*V, g)$ has type $(p-m, m, \mathbf{o}')$ for some \mathbf{o}' , and the rank is $(p-m) \text{rank } V/p$.*

Proof We have only to consider the case that (\mathcal{P}_*V, g) has type (p, m, \mathbf{o}) . Let $\varphi_p : U_\eta \rightarrow U_\tau$ be given by $\varphi_p(\eta) = \eta^p$. Let $\varphi : U_u \rightarrow U_\zeta$ be given by $\varphi(u) = u^{p-m}$. Let \mathcal{P}_*V be the filtered bundle on U_u obtained as the pull back of $\mathcal{N}_*^{\infty,0}(\mathcal{P}_*V, g)$ by φ .

We use the decomposition $\varphi_p^*(\mathcal{P}_*V, g) = \bigoplus_{\alpha \in \mathbf{o}} (\mathcal{P}_*V_\alpha^{(p)}, g_\alpha^{(p)})$. We consider the following complex on $U_{\eta, \zeta}$:

$$\pi_1^* \mathcal{P}_{pc} V_\alpha^{(p)} \xrightarrow{g_\alpha^{(p)} - \zeta} \pi_1^* \mathcal{P}_{pc} V_\alpha^{(p)}$$

The quotient is denoted by \mathcal{M}'_c . We have $\pi_{2*} \mathcal{M}'_c \simeq \mathcal{N}_{\kappa_2(p,m,c)+1}^{\infty,0}(\mathcal{P}_*V, g)$. Because $g_\alpha^{(p)}(\mathcal{P}_a V_\alpha^{(p)}) \subset \mathcal{P}_a V_{<\alpha}^{(p)}$, we have the following exact sequence, as in the case of $\mathcal{N}^{0,\infty}$ (see the proof of Lemma 5.3):

$$0 \longrightarrow \pi_1^* \text{Gr}_{pc}^{\mathcal{P}}(V_\alpha^{(p)}) \xrightarrow{-\zeta} \pi_1^* \text{Gr}_{pc}^{\mathcal{P}}(V_\alpha^{(p)}) \longrightarrow \mathcal{M}'_c / \mathcal{M}'_{<c} \longrightarrow 0$$

It induces the following isomorphism of \mathbb{C} -vector spaces:

$$\text{Gr}_{pc}^{\mathcal{P}}(V_\alpha^{(p)}) \simeq \frac{\mathcal{N}_{\kappa_2(p,m,c)+1}^{\infty,0}(\mathcal{P}_*V, g)}{\mathcal{N}_{<\kappa_2(p,m,c)+1}^{\infty,0}(\mathcal{P}_*V, g)} \quad (51)$$

We take a frame \mathbf{v} of $\mathcal{P}_{pc}V_\alpha^{(p)}$ compatible with the parabolic structure. We set $\nu_{ij} := \eta^i v_j$. By the isomorphism (51), they induce a frame of $\mathcal{N}_{\kappa_2(p,c,m)+1}^{\infty,0}(\mathcal{P}_*V, g)$ compatible with the parabolic structure. We set $\tilde{\nu}_{ij} := u^{-i}\eta^i v_j$. The tuple $\tilde{\nu}$ induces a frame of $\mathcal{P}_{p(c+1)-m/2}\mathcal{V}$ compatible with the parabolic structure.

We set $h := \eta^{m-p}g_\alpha^{(p)}$ on $\mathcal{P}_{pc}V_\alpha^{(p)}$, which is invertible. We have $\eta^{-p+m}u^{p-m} = h$ on \mathcal{V} . Let k be the integer determined by the condition $0 \leq -p + k(p-m) < p-m$. We set $a := -p + k(p-m)$. We have $\eta^{-p}u^p = \eta^a u^{-a} h^k = \eta^{a-(p-m)} u^{-a+(p-m)} h^{k-1}$. We have

$$u^p \eta^{-p} \nu_{ij} = \begin{cases} \eta^{a+i} u^{-(a+i)} h^k (v_j) & (a+i < p-m) \\ \eta^{a+i-(p-m)} u^{-(a+i)+p-m} h^{k-1} (v_j) & (a+i \geq p-m) \end{cases}$$

Hence, $u^p \eta^{-p}$ preserves $\mathcal{P}_{p(c+1)-m/2}\mathcal{V}$.

By the frames \mathbf{v} and $\tilde{\mathbf{v}}$, we have an isomorphism $\mathcal{P}_{p(c+1)-m/2}\mathcal{V}_{|0}$ and $\mathcal{P}_{pc}V_{\alpha|0} \otimes \mathbb{C}^{p-m}$. We take a refinement \tilde{F} of the parabolic filtration of $\mathcal{P}_{pc}V_{\alpha|0}$ such that (i) \tilde{F} is preserved by $h_{|0}$, (ii) the induced endomorphism on $\text{Gr}^{\tilde{F}}$ is semisimple with a unique eigenvalue β . It induces a filtration \tilde{F}' of $\mathcal{P}_{p(c+1)-m/2}\mathcal{V}_{|0}$. On $\text{Gr}^{\tilde{F}'}$, $u^p \eta^{-p}$ is expressed by the matrix

$$\sum E_{a+i,i} \otimes \beta^k I + \sum E_{i,i+p-m-a} \otimes \beta^{k-1} I$$

with respect to an appropriate base. Then, we obtain that $(\mathcal{P}_*\mathcal{N}^{\infty,0}(\mathcal{P}_*V, g), \theta)$ has type $(p-m, m, \mathbf{o}')$ for some \mathbf{o}' . ■

Corollary 5.6 *The construction $\mathcal{N}_*^{\infty,0}$ gives a functor from the category of the germs of admissible filtered Higgs bundles satisfying (C0) to the category of the germs of admissible filtered Higgs bundles.* ■

We denote $(\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta), g)$ in §5.1.3 by $\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta)$ for simplicity. We also denote $(\mathcal{N}_*^{\infty,0}(\mathcal{P}_*V, g), \theta)$ by $\mathcal{N}_*^{\infty,0}(\mathcal{P}_*V, g)$.

Proposition 5.7

- Suppose that (\mathcal{P}_*V, θ) is admissible such that $V_0^{(1,0)} = 0$ in the type decomposition. Then, we have a natural isomorphism $\mathcal{N}_*^{\infty,0}\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta) \simeq (\mathcal{P}_*V, \theta)$.
- Suppose that (\mathcal{P}_*V, g) is admissible and satisfies the condition (C0). Then, we have a natural isomorphism $\mathcal{N}_*^{0,\infty}\mathcal{N}_*^{\infty,0}(\mathcal{P}_*V, g) \simeq (\mathcal{P}_*V, g)$.

Proof Suppose that (\mathcal{P}_*V, θ) has type (p, m) . Note that, if we set $d := \kappa_1(p, m, c)$, then we have $\kappa_2(p + m, m, d) = c$. Let p_i be the projection of $U_\zeta \times U_\tau \times U_{\zeta'}$ onto the i -th component. We have the following diagram on $U_\zeta \times U_\tau \times U_{\zeta'}$:

$$\begin{array}{ccc} p_1^* \mathcal{P}_{c-m/p}(V) & \xrightarrow{\tau\theta+d\zeta} & p_1^* \mathcal{P}_c(V) d\zeta/\zeta \\ \zeta-\zeta' \downarrow & & \zeta-\zeta' \downarrow \\ p_1^* \mathcal{P}_{c-m/p}(V) & \xrightarrow{\tau\theta+d\zeta} & p_1^* \mathcal{P}_c(V) d\zeta/\zeta \end{array}$$

We regard it as a double complex, where the left upper $p_1^* \mathcal{P}_{c-m/p}(V)$ sits in the degree $(0, 0)$. Let C^\bullet denote the associated total complex. By the construction, $\mathcal{N}_{c+1}^{\infty,0}\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta)$ is obtained as $p_{3*}\mathcal{H}^2(C^\bullet)$. We can observe that it is isomorphic to the push-forward of \mathcal{Q}_c in §5.1.3 by the projection $U_{\zeta,\tau} \rightarrow U_\zeta$, which is naturally isomorphic to $\mathcal{P}_c V d\zeta/\zeta \simeq \mathcal{P}_{c+1}V$. The action of $-\tau^{-1}$ is equal to f for the expression $\theta = f d\zeta$. Hence, we obtain the desired isomorphism $\mathcal{N}_*^{\infty,0}\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta) \simeq (\mathcal{P}_*V, \theta)$.

Suppose that (\mathcal{P}_*V, g) has type (p, m) with $p > m$. Let p_i denote the projection of $U_\tau \times U_\zeta \times U_{\tau'}$ onto the i -th component. We have the following commutative diagram of the sheaves on $U_\tau \times U_\zeta \times U_{\tau'}$:

$$\begin{array}{ccc} p_1^* \mathcal{P}_{c-1}V & \xrightarrow{g-\zeta} & p_1^* \mathcal{P}_{c-1}V \\ (-\tau'(\tau^{-1})+1)d\zeta \downarrow & & \downarrow (-\tau'(\tau^{-1})+1)d\zeta \\ p_1^* \mathcal{P}_c V d\zeta & \xrightarrow{g-\zeta} & p_1^* \mathcal{P}_c V d\zeta \end{array}$$

We regard it as the double complex, where the left upper $p_1^* \mathcal{P}_{c-1} V$ sits in the degree $(0, 0)$. Let C^\bullet denote the associated total complex. By the construction, $\mathcal{N}_c^{0, \infty} \mathcal{N}_*^{\infty, 0}(\mathcal{P}_* V, g)$ is naturally isomorphic to $p_{3*} \mathcal{H}^2(C^\bullet)$. We can observe that it is naturally isomorphic to the push-forward of \mathcal{M}_c in §5.1.4 by the projection $U_\tau \times U_\zeta \rightarrow U_\tau$, which is naturally isomorphic to $\mathcal{P}_c V$. The action of ζ is given by g . Hence, we obtain the desired isomorphism $\mathcal{N}_*^{0, \infty} \mathcal{N}_*^{\infty, 0}(\mathcal{P}_* V, g) \simeq (\mathcal{P}_* V, g)$. \blacksquare

5.1.5 Description of the functors

Let $(\mathcal{P}_* V, \theta)$ be a filtered Higgs bundle with slope $(p, m) \neq (1, 0)$ on U_ζ . Suppose that there exist a ramified covering $\varphi_q : U_{\zeta_q} \rightarrow U_\zeta$ and a filtered Higgs bundle $(\mathcal{P}_* V', \theta')$ on U_{ζ_q} with an isomorphism $\varphi_{q*}(\mathcal{P}_* V', \theta') \simeq (\mathcal{P}_* V, \theta)$. For $c \in \mathbb{R}$, we consider the following morphism on $U_{\zeta_q, \tau}$:

$$\mathcal{P}_{q(c-m/p)} V' \xrightarrow{\tau^{\theta'} + d\zeta_q^q} \mathcal{P}_{qc} V' d\zeta_q / \zeta_q$$

The quotient is denoted \mathcal{Q}'_c . The following lemma is clear by construction.

Lemma 5.8 $\pi_{2*} \mathcal{Q}'_c$ is naturally isomorphic to $\mathcal{N}_{\kappa_1(p, m, c)}^{0, \infty}(\mathcal{P}_* V, \theta)$. \blacksquare

Let $(\mathcal{P}_* V, \psi)$ be a filtered Higgs bundle with slope (p, m) on U_τ , such that $(p, m) \neq (1, 0)$ and $p > m$. Suppose that there exist a ramified covering $\varphi_q : U_{\tau_q} \rightarrow U_\tau$ and a filtered Higgs bundle $(\mathcal{P}_{q*} V', \psi')$ on U_{τ_q} with an isomorphism $\varphi_{q*}(\mathcal{P}_{q*} V', \psi') \simeq (\mathcal{P}_* V, \psi)$. Let $\psi' = g' \varphi^*(-\tau^{-2} d\tau)$. For $c \in \mathbb{R}$, we consider the following morphism on $U_{\tau_q, \zeta}$:

$$\mathcal{P}_{qc} V' \xrightarrow{g' - \zeta} \mathcal{P}_{qc} V'$$

Let \mathcal{M}'_c denote the quotient. The following is clear by construction.

Lemma 5.9 $\pi_{2*} \mathcal{M}'_c$ is naturally isomorphic to $\mathcal{N}_{\kappa_2(p, m, c)+1}^{\infty, 0}(\mathcal{P}_* V, g)$. \blacksquare

5.2 Algebraic Nahm transform for admissible filtered Higgs bundle

5.2.1 Transform

Let $T^\vee := \mathbb{C}/L^\vee$. Let $D \subset T^\vee$ be an effective reduced divisor. Let $(\mathcal{P}_* \mathcal{E}, \theta)$ be a filtered Higgs bundle on (T^\vee, D) . Suppose that it is admissible around each point of D in the sense of §5.1.1. We shall construct an object $N(\mathcal{P}_* \mathcal{E}, \theta)$ in $D^b(\mathcal{O}_{T \times \mathbb{P}^1})$.

For $I \subset \{1, 2, 3\}$, let p_I be the projections of $T^\vee \times T \times \mathbb{P}^1$ onto the product of the i -th components ($i \in I$). Let \mathcal{Poin} be the Poincaré bundle on $T^\vee \times T$. Applying the construction in §5.1.2 around each point of D , we extend \mathcal{E} and $\mathcal{E} \otimes \Omega_X^1$ on $X \setminus D$ to $\mathcal{C}^0(\mathcal{P}_* \mathcal{E}, \theta)$ and $\mathcal{C}^1(\mathcal{P}_* \mathcal{E}, \theta)$, respectively. We set

$$\tilde{\mathcal{C}}^0(\mathcal{P}_* \mathcal{E}, \theta) := p_1^* \mathcal{C}^0(\mathcal{P}_* \mathcal{E}, \theta) \otimes p_{12}^* \mathcal{Poin} \otimes p_3^* \mathcal{O}_{\mathbb{P}^1}(-1), \quad \tilde{\mathcal{C}}^1(\mathcal{P}_* \mathcal{E}, \theta) := p_1^* \mathcal{C}^1(\mathcal{P}_* \mathcal{E}, \theta) \otimes p_{12}^* \mathcal{Poin}.$$

Let ζ be the standard coordinate of \mathbb{C} , which induces local coordinates of T^\vee . We have the holomorphic 1-form $d\zeta$ on T^\vee . Let w be the standard coordinate of $\mathbb{C} \subset \mathbb{P}^1$, which we can naturally regard as a section of $\mathcal{O}_{\mathbb{P}^1}(1)$. Then, we have the following morphism:

$$\theta + w d\zeta : \tilde{\mathcal{C}}^0(\mathcal{P}_* \mathcal{E}, \theta) \rightarrow \tilde{\mathcal{C}}^1(\mathcal{P}_* \mathcal{E}, \theta). \quad (52)$$

Thus, we obtain a complex $\tilde{\mathcal{C}}^\bullet(\mathcal{P}_* \mathcal{E}, \theta)$ on $T^\vee \times T \times \mathbb{P}^1$. We define

$$N(\mathcal{P}_* \mathcal{E}, \theta) := R p_{23*}(\tilde{\mathcal{C}}^\bullet(\mathcal{P}_* \mathcal{E}, \theta))[1].$$

Lemma 5.10 *There exists a neighbourhood U of ∞ in \mathbb{P}^1 , such that $\mathcal{H}^i(N(\mathcal{P}_* \mathcal{E}, \theta))|_{T \times U} = 0$ unless $i \neq 0$. Moreover, $\mathcal{H}^0(N(\mathcal{P}_* \mathcal{E}, \theta))|_{T \times \{\infty\}}$ is semistable of degree 0.*

Proof Let π_i denote the projection of $T^\vee \times \mathbb{P}^1$ onto the i -th component. We have the following complex $\tilde{\mathcal{C}}_1^\bullet(\mathcal{P}_*\mathcal{E}, \theta)$ on $T^\vee \times \mathbb{P}^1$:

$$\pi_1^* \mathcal{C}^0(\mathcal{P}_*\mathcal{E}, \theta) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\theta + w d\zeta} \pi_1^* \mathcal{C}^1(\mathcal{P}_*\mathcal{E}, \theta)$$

By the construction, $N(\mathcal{P}_*\mathcal{E}, \theta)$ is isomorphic to $\widehat{\text{RFM}}_+(\tilde{\mathcal{C}}_1^\bullet(\mathcal{P}_*\mathcal{E}, \theta))[1]$. If U is sufficiently small, $\theta + w d\zeta$ is injective on $T^\vee \times U$, and the support of the cokernel is relatively 0-dimensional over U . Then, the claim of the lemma follows. \blacksquare

We impose the following condition.

(A0) $\mathcal{H}^i(N(\mathcal{P}_*\mathcal{E}, \theta)) = 0$ unless $i = 0$.

Under the assumption, we naturally identify $N(\mathcal{P}_*\mathcal{E}, \theta)$ with the 0-th cohomology sheaf $\mathcal{H}^0(N(\mathcal{P}_*\mathcal{E}, \theta))$, and it is a vector bundle on $T \times \mathbb{P}^1$. We define

$$\text{Nahm}(\mathcal{P}_*\mathcal{E}, \theta) := N(\mathcal{P}_*\mathcal{E}, \theta) \otimes \mathcal{O}_{T \times \mathbb{P}^1}(* (T \times \{\infty\})).$$

We shall define a filtered bundle $\text{Nahm}_*(\mathcal{P}_*\mathcal{E}, \theta)$ over $\text{Nahm}(\mathcal{P}_*\mathcal{E}, \theta)$.

By Lemma 5.10, there exists a neighbourhood U of $\infty \in \mathbb{P}^1$ such that $N(\mathcal{P}_*\mathcal{E}, \theta)|_{T \times \{\tau_1\}}$ are semistable of degree 0 for any $\tau_1 \in U$. Let $\tilde{\mathfrak{s}} \subset T^\vee \times U$ denote the spectrum. We have $\tilde{\mathfrak{s}} \cap (T^\vee \times \{\infty\}) \subset D$. We fix a lift of D to $\tilde{D} \subset \mathbb{C}$. Then, if U is sufficiently small, we may have a lift of \mathfrak{s} to $\tilde{\mathfrak{s}} \subset \mathbb{C} \times U$. We obtain the corresponding holomorphic vector bundle V with an endomorphism g such that $\mathcal{S}p(g) \subset \tilde{\mathfrak{s}}$. (See §2.1.3.) We have the decomposition

$$(V, g) = \bigoplus_{P \in D} (V_P, g_P),$$

where $\mathcal{S}p(g_P) \cap (\mathbb{C} \times \{\infty\})$ is the lift of P . We have the induced decomposition on $T \times U$:

$$\text{Nahm}(\mathcal{P}_*\mathcal{E}, \theta) = \bigoplus_{P \in D} \text{Nahm}(\mathcal{P}_*\mathcal{E}, \theta)_P$$

Let $U_P \subset T^\vee$ be a small neighbourhood of $P \in D$. We use the coordinate $\zeta_P := \zeta - \tilde{P}$. By the construction, we have a natural isomorphism $V_P \simeq \mathcal{N}^{0, \infty}(\mathcal{P}_*(\mathcal{E}, \theta)|_{U_P})$. We have $g_P = g'_P + \tilde{P} \text{ id}$, where g'_P is the endomorphism induced by ζ_P . Thus, we obtain a filtered bundle $\text{Nahm}_*(\mathcal{P}_*\mathcal{E}, \theta)_P$ over $\text{Nahm}(\mathcal{P}_*\mathcal{E}, \theta)_P$, by transferring $\mathcal{N}^{0, \infty}(\mathcal{P}_*(\mathcal{E}, \theta)|_{U_P})$. By taking the direct sum, we obtain a filtered bundle $\text{Nahm}_*(\mathcal{P}_*\mathcal{E}, \theta)$ over $\text{Nahm}(\mathcal{P}_*\mathcal{E}, \theta)$.

Remark 5.11 We obtain a different transformation by replacing Poin and $w d\zeta$ with Poin^\vee and $-w d\zeta$, respectively, for which we can argue in a similar way. \blacksquare

Remark 5.12 In [9], the Fourier transform for Higgs bundles on smooth projective curves are studied. The algebraic Nahm transform in this paper can be regarded as a filtered variant, although we consider only the case where the base space is an elliptic curve. \blacksquare

5.2.2 Characteristic number

For compact complex manifolds Z_i ($i = 1, 2$), let $\omega_{Z_i} \in H^*(Z_1 \times Z_2)$ denote the pull back of the fundamental class of Z_i by the projection.

Lemma 5.13 We have $\int_{T \times \mathbb{P}^1} c_1(\text{Nahm}_a(\mathcal{P}_*\mathcal{E}, \theta)) \omega_{\mathbb{P}^1} = 0$ for any $a \in \mathbb{R}$.

Proof It follows from that $\text{Nahm}_a(\mathcal{P}_*\mathcal{E}, \theta) / \text{Nahm}_{<a}(\mathcal{P}_*\mathcal{E}, \theta)$ is of degree 0 for any $a \in \mathbb{R}$. \blacksquare

The following lemma can be checked easily.

Lemma 5.14 $c_2(\text{Nahm}_a(\mathcal{P}_*\mathcal{E}, \theta))$ is independent of $a \in \mathbb{R}$. We denote it by $c_2(\text{Nahm}_*(\mathcal{P}_*\mathcal{E}, \theta))$. \blacksquare

Around each $P \in D$, we have the type decomposition $(\mathcal{P}_*\mathcal{E}, \theta) = \bigoplus_{(p,m,\mathbf{o})} (\mathcal{P}_*\mathcal{E}_{\mathbf{o}}^{(p,m)}, \theta_{P,\mathbf{o}}^{(p,m)})$. We set

$$\ell_P := \dim \text{Cok} \left(\text{Res}(\theta) : \text{Gr}_0^{\mathcal{P}}(\mathcal{E}_{P,0}^{(1,0)}) \longrightarrow \text{Gr}_0^{\mathcal{P}}(\mathcal{E}_{P,0}^{(1,0)}) \right).$$

We put $r_{P,\mathbf{o}}^{(p,m)} = \text{rank}(\mathcal{E}_{P,\mathbf{o}}^{(p,m)})/p$ and $r_P^{(p,m)} := \sum_{\mathbf{o}} r_{P,\mathbf{o}}^{(p,m)}$. We have $\sum_{p,m} r_P^{(p,m)} p = \text{rank } \mathcal{E}$.

Proposition 5.15 *We have the following equalities:*

$$\text{rank Nahm}(\mathcal{P}_*\mathcal{E}, \theta) = \sum_P \sum_{p,m} r_P^{(p,m)} (p+m) - \sum_P \ell_P \quad (53)$$

$$\int_{T \times \mathbb{P}^1} c_1(\text{Nahm}_*(\mathcal{P}_*\mathcal{E}, \theta)) \cdot \omega_T = \deg(\mathcal{P}_*\mathcal{E}) \quad (54)$$

$$\int_{T \times \mathbb{P}^1} c_2(\text{Nahm}_*(\mathcal{P}_*\mathcal{E}, \theta)) = \text{rank } \mathcal{E} \quad (55)$$

Proof Let us prove (53) and (54). We have only to consider the rank and the degree of $\text{Nahm}_*(\mathcal{P}_*\mathcal{E}, \theta)|_{\{0\} \times \mathbb{P}^1}$. Let $\mathcal{V} \subset \mathcal{P}_1\mathcal{E}$ be the subsheaf determined by the following conditions:

- $\mathcal{V} = \mathcal{P}_1\mathcal{E}$ on the complement of D .
- It has a decomposition $\mathcal{V} = \bigoplus_P \mathcal{V}_{P,\mathbf{o}}^{(p,m)}$ around each $P \in D$.
- We have $\mathcal{V}_{P,\mathbf{o}}^{(p,m)} = \mathcal{P}_{1/2}\mathcal{E}_{P,\mathbf{o}}^{(p,m)}$ for $(p,m,\mathbf{o}) \neq (1,0,0)$, and $\mathcal{V}_{P,0}^{(1,0)} = \mathcal{P}_1\mathcal{E}_{P,0}^{(1,0)}$.

Let π_i denote the projection of $T^\vee \times \mathbb{P}^1$ onto the i -th component. We have the following K -theoretic description:

$$\left(\tilde{\mathcal{C}}^1(\mathcal{P}_*\mathcal{E}, \theta) - \tilde{\mathcal{C}}^0(\mathcal{P}_*\mathcal{E}, \theta) \right)_{|T^\vee \times \{0\} \times \mathbb{P}^1} = \pi_1^* \left(\mathcal{V} - \sum_{P \in D} \mathcal{O}_P^{\oplus \ell_P} \right) - \pi_2^* \mathcal{O}_{\mathbb{P}^1}(-1) \cdot \pi_1^* \left(\mathcal{V} - \sum_{P \in D} \sum_{p,m,\mathbf{o}} \mathcal{O}_P^{\oplus r_{P,\mathbf{o}}^{(p,m)}(p+m)} \right) \quad (56)$$

The Chern character of (56) is equal to the following:

$$\begin{aligned} \pi_1^* \text{ch}(\mathcal{V}) - \sum_{P \in D} \ell_P \omega_{T^\vee} - (1 - \omega_{\mathbb{P}^1}) \left(\pi_1^* \text{ch}(\mathcal{V}) - \sum_P \sum_{p,m} r_P^{(p,m)} (p+m) \omega_{T^\vee} \right) \\ = \left(\sum_P \sum_{p,m} r_P^{(p,m)} (p+m) - \sum_P \ell_P \right) \omega_{T^\vee} + \omega_{\mathbb{P}^1} \pi_1^* \text{ch}(\mathcal{V}) - \omega_{\mathbb{P}^1} \sum_P \sum_{p,m} r_P^{(p,m)} (p+m) \omega_{T^\vee} \end{aligned} \quad (57)$$

Hence, the Chern character of $N(\mathcal{P}_*\mathcal{E}, \theta)|_{\{0\} \times \mathbb{P}^1}$ is

$$\begin{aligned} \sum_P \sum_{p,m} r_P^{(p,m)} (p+m) - \sum_P \ell_P + \omega_{\mathbb{P}^1} \left(\deg(\mathcal{V}) - \sum_P \sum_{p,m} r_P^{(p,m)} (p+m) \right) \\ = \sum_P \sum_{p,m} r_P^{(p,m)} (p+m) - \sum_P \ell_P + \omega_{\mathbb{P}^1} \left(\deg(\mathcal{V}(-D)) - \sum_P \sum_{p,m} r_P^{(p,m)} m \right). \end{aligned} \quad (58)$$

In particular, we obtain (53). We also obtain the following equality:

$$\deg(N(\mathcal{P}_*\mathcal{E}, \theta)|_{\{0\} \times \mathbb{P}^1}) = \deg(\mathcal{V}(-D)) - \sum_P \sum_{p,m} r_P^{(p,m)} m$$

For the parabolic characteristic numbers, we have the following expressions:

$$\deg(\mathcal{P}_*\mathcal{E}) = \deg(\mathcal{V}(-D)) - \sum_{P \in D} \sum_{p,m,\mathbf{o}} \delta(\mathcal{P}_*\mathcal{E}_{P,\mathbf{o}}^{(p,m)})$$

$$\deg(\text{Nahm}_*(\mathcal{P}_*\mathcal{E}, \theta)|_{\{0\} \times \mathbb{P}^1}) = \deg(\mathbf{N}(\mathcal{P}_*\mathcal{E}, \theta)|_{\{0\} \times \mathbb{P}^1}) - \sum_{P \in D} \sum_{p, m, \mathbf{o}} \delta(\text{Nahm}_*(\mathcal{P}_*\mathcal{E}, \theta)_{P, \mathbf{o}}^{(p+m, m)})$$

Let us look at the contribution of the parabolic structure. In the case $(p, m, \mathbf{o}) = (1, 0, 0)$, we have

$$\delta(\text{Nahm}_*(\mathcal{P}_*\mathcal{E}, \theta)_{P, 0}^{(1, 0)}) = \sum_{-1 < c < 0} c \dim \text{Gr}_c^{\mathcal{P}} \text{Nahm}(\mathcal{P}_*\mathcal{E}, \theta)_{P, 0}^{(1, 0)} = \sum_{-1 < c < 0} c \dim \text{Gr}_c^{\mathcal{P}} (\mathcal{E}_{P, 0}^{(1, 0)}) = \delta(\mathcal{P}_*\mathcal{E}_{P, 0}^{(1, 0)}).$$

Let us consider the case $(p, m, \mathbf{o}) \neq (1, 0, 0)$. Let $\varphi_p : U_u \rightarrow U_P$ be given by $\varphi(u) = u^p$. We have the decomposition $\varphi^*(\mathcal{P}_*\mathcal{E}_{P, \mathbf{o}}^{(p, m)}, \theta_{P, \mathbf{o}}^{(p, m)}) = \bigoplus_{\alpha \in \mathbf{o}} (\mathcal{P}_*V_\alpha, \theta_\alpha)$. For any $c \in \mathbb{R}$, we put

$$r_{P, \mathbf{o}, c}^{(p, m)} := \dim \text{Gr}_c^{\mathcal{P}} V_\alpha.$$

We have the following equality:

$$\delta(\mathcal{P}_*\mathcal{E}_{O, \mathbf{o}}^{(p, m)}) = \sum_{\substack{-p/2-1 < c \leq -p/2 \\ 0 \leq j \leq p-1}} r_{P, \mathbf{o}, c}^{(p, m)} \frac{c-j}{p} = \sum_{-p/2-1 < c \leq -p/2} r_{P, \mathbf{o}, c}^{(p, m)} \left(c - \frac{1}{2}(p-1) \right)$$

We also have the following equality from the expression of the parabolic structure of $\mathcal{N}_*^{0, \infty}(\mathcal{P}_*\mathcal{E}, \theta)$ in the proof of Lemma 5.3:

$$\delta(\text{Nahm}(\mathcal{P}_*\mathcal{E}, \theta)_{P, \mathbf{o}}^{(p+m, m)}) = \sum_{\substack{-p/2-1 < c \leq -p/2 \\ 0 \leq j \leq p+m-1}} r_{P, \mathbf{o}, c}^{(p, m)} \frac{2c-2j-m}{2(p+m)} = \sum_{-p/2-1 < c \leq -p/2} r_{P, \mathbf{o}, c}^{(p, m)} \left(c - m - \frac{1}{2}(p-1) \right)$$

Then, the equality (54) follows from $\sum_{c, \mathbf{o}} r_{P, c, \mathbf{o}}^{(p, m)} = r_P^{(p, m)}$.

Let us prove (55). We have $\int_{T \times \mathbb{P}^1} c_2(\text{Nahm}_*(\mathcal{P}_*\mathcal{E}, \theta)) = \int_{T \times \mathbb{P}^1} c_2(\mathbf{N}(\mathcal{P}_*\mathcal{E}, \theta))$. We also have the following:

$$\int_{T \times \mathbb{P}^1} c_2(\mathbf{N}(\mathcal{P}_*\mathcal{E}, \theta)) = - \int_{T \times \mathbb{P}^1} \text{ch}_2(\mathbf{N}(\mathcal{P}_*\mathcal{E}, \theta)) = - \int_{T^\vee \times T \times \mathbb{P}^1} \text{ch}_3(\tilde{\mathcal{C}}^1 - \tilde{\mathcal{C}}^0)$$

We have $\text{ch}_3(\tilde{\mathcal{C}}^1) = 0$. We have $c_1(\mathcal{P}oin)^2 = -2\omega_T\omega_{T^\vee}$. We also have

$$\int_{T^\vee \times T \times \mathbb{P}^1} \text{ch}_3(\tilde{\mathcal{C}}^0) = \int_{T^\vee \times T \times \mathbb{P}^1} \text{rank}(\mathcal{V})\omega_T\omega_{T^\vee}\omega_{\mathbb{P}^1} = \text{rank}(\mathcal{V}).$$

Hence, we obtain (55). ■

5.3 Algebraic Nahm transform for admissible filtered bundles

5.3.1 Preliminary

Let $U \subset \mathbb{P}^1$ be a small neighbourhood of ∞ . We introduce some conditions on filtered bundles \mathcal{P}_*E on $(T \times U, T \times \{\infty\})$.

(A1) $\mathcal{P}_cE|_{T \times \infty}$ are semistable of degree 0 for any $c \in \mathbb{R}$.

Let $\mathcal{S}p_\infty(E) \subset T^\vee$ denote the spectrum of $\mathcal{P}_cE|_{T \times \infty}$. It is independent of c . We fix its lift to $\widetilde{\mathcal{S}p}_\infty(E) \subset \mathbb{C}$. Then, for a small neighbourhood U' of $\infty \in \mathbb{P}^1$, we obtain the corresponding filtered bundle \mathcal{P}_*V with an endomorphism g on (U', ∞) such that $\mathcal{S}p(g|_\infty) = \widetilde{\mathcal{S}p}_\infty(E)$. We have the decomposition

$$(\mathcal{P}_*V, g) = \bigoplus_{P \in \mathcal{S}p_\infty(E)} (\mathcal{P}_*V_P, g_P)$$

such that $\mathcal{S}p(g_P) \cap (\mathbb{C} \times \{\infty\})$ is the lift of P . A filtered bundle satisfying (A1) is called admissible, if it satisfies the following condition, which is independent of the choice of $\widetilde{\mathcal{S}p}_\infty(E)$.

(A2) $(\mathcal{P}_*V_P, g_P - \tilde{P} \text{ id})$ is admissible in the sense of §5.1.4 for any P .

We introduce a stability condition for filtered bundles on $(T \times \mathbb{P}^1, T \times \{\infty\})$ satisfying the conditions (A1,2), by following [8]. Let $\omega_T \in H^2(T \times \mathbb{P}^1, \mathbb{Z})$ denote the pull back of the fundamental class of T by the projection $T \times \mathbb{P}^1 \rightarrow T$. For a filtered bundle $\mathcal{P}_*\mathcal{E}$ on $(T \times \mathbb{P}^1, T \times \{\infty\})$, we define the degree of $\mathcal{P}_*\mathcal{E}$ by

$$\deg(\mathcal{P}_*\mathcal{E}) := \int_{T \times \mathbb{P}^1} \text{par-c}_1(\mathcal{P}_*\mathcal{E})\omega_T = \int_{\{z\} \times \mathbb{P}^1} \text{par-c}_1(\mathcal{P}_*\mathcal{E}).$$

We say that the filtered bundle \mathcal{P}_*E is stable (semistable), if $\deg(\mathcal{P}_*\mathcal{E}) < 0$ (resp. $\deg(\mathcal{P}_*\mathcal{E}) \leq 0$) for any $\mathcal{P}_*\mathcal{E} \subset \mathcal{P}_*E$ such that (i) $0 < \text{rank } \mathcal{E} < \text{rank } E$, (ii) $\mathcal{P}_*\mathcal{E}$ also satisfies (A1,2). We say that a semistable filtered bundle \mathcal{P}_*E is polystable, if it has a decomposition $\mathcal{P}_*E = \bigoplus \mathcal{P}_*E_i$ such that each \mathcal{P}_*E_i is stable. The following lemma is clear.

Lemma 5.16 *Let \mathcal{P}_*E be a filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$ satisfying (A1,2). If \mathcal{P}_*E is stable, then \mathcal{P}_*E^\vee is also stable.* ■

It is standard that the stability condition implies the vanishing of some cohomology groups.

Lemma 5.17 *Let \mathcal{P}_*E be a filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$ satisfying (A1,2). If \mathcal{P}_*E is stable, it satisfies the following condition (A3).*

(A3) For $n = 0, 1$ and for any $L \in T^\vee$, we have $H^i(T \times \mathbb{P}^1, \mathcal{P}_nE \otimes L) = 0$ unless $i = 1$.

Proof Because \mathcal{P}_*E is stable of degree 0, we obtain that $H^0(T \times \mathbb{P}^1, \mathcal{P}_cE) = 0$ for any $c \leq 0$. Indeed, a non-zero section induces a filtered subsheaf $\mathcal{P}_*\mathcal{O} \subset \mathcal{P}_*E$, which is strict with $\deg(\mathcal{P}_*\mathcal{O}) \geq 0$. Because $(\mathcal{P}_*E)^\vee$ is also stable of degree 0, we have $H^0(T \times \mathbb{P}^1, \mathcal{P}_cE^\vee) = 0$ for any $c \leq 0$. By using the Serre duality, we obtain that $H^2(T \times \mathbb{P}^1, \mathcal{P}_cE) = 0$ for $c \geq -1$. Hence, \mathcal{P}_*E satisfies the conditions (A3). ■

We give a simple remark on some cohomology group associated to \mathcal{P}_*E satisfying (A1).

Lemma 5.18 *Let \mathcal{P}_*E be a filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$ satisfying (A1). Let L be a holomorphic line bundle on T . Suppose $0 \notin \text{Sp}_\infty(\mathcal{P}_*E \otimes L)$. Let \mathcal{U} be any $\mathcal{O}_{T \times \mathbb{P}^1}$ -submodule of \mathcal{P}_cE ($c \in \mathbb{R}$) such that (i) $\mathcal{U}|_{T \times \mathbb{C}_w} = \mathcal{P}_cE|_{T \times \mathbb{C}_w}$, (ii) $\mathcal{U}|_{T \times \{\infty\}}$ is semistable of degree 0. Then, we have $H^j(T \times \mathbb{P}^1, \mathcal{U} \otimes L) = H^j(T \times \mathbb{P}^1, \mathcal{P}_cE \otimes L)$.*

Proof Let $\pi : T \times \mathbb{P}^1 \rightarrow T$ be the projection. By the assumption, we obtain $R\pi_*(\mathcal{U} \otimes L) \simeq R\pi_*(\mathcal{P}_cE \otimes L)$, because both of them vanish around ∞ . Then, the claim of the lemma follows. ■

5.3.2 Transform to filtered Higgs bundles

For $I \subset \{1, 2, 3\}$, let p_I be the projection of $T^\vee \times T \times \mathbb{P}^1$ onto the product of the i -th components ($i \in I$). Let Poin denote the Poincaré bundle on $T^\vee \times T$.

Let \mathcal{P}_*E be a filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$ satisfying the conditions (A1–3). We put $D := \text{Sp}_\infty(\mathcal{P}_*E)$. We define

$$\text{Nahm}(\mathcal{P}_*E) := R^1p_{1*}(p_{12}^*\text{Poin}^\vee \otimes p_{23}^*\mathcal{P}_{-1}E) \otimes \mathcal{O}_{T^\vee}(*D).$$

By (A3), $\text{Nahm}(\mathcal{P}_*E)$ is a locally free $\mathcal{O}_{T^\vee}(*D)$ -module. By Lemma 5.18, we have a natural isomorphism

$$\text{Nahm}(\mathcal{P}_*E) \simeq R^1p_{1*}(p_{12}^*\text{Poin}^\vee \otimes p_{23}^*\mathcal{P}_0E) \otimes \mathcal{O}_{T^\vee}(*D).$$

Let w be the standard coordinate of $\mathbb{C} \subset \mathbb{P}^1$. It naturally gives a section of $\mathcal{O}_{\mathbb{P}^1}(1)$. The multiplication of w induces an endomorphism g of $\text{Nahm}(\mathcal{P}_*E)$. We obtain a Higgs field $\theta := -gd\zeta$ of $\text{Nahm}(\mathcal{P}_*E)$. We shall define a filtered bundle $\text{Nahm}_*(\mathcal{P}_*E) = (\text{Nahm}_a(\mathcal{P}_*E) \mid a \in \mathbb{R})$ over $\text{Nahm}(\mathcal{P}_*E)$.

We have the decomposition $\mathcal{P}_*E = \bigoplus_{P \in \text{Sp}_\infty(E)} \bigoplus_{p,m,o} \mathcal{P}_*E_{P,o}^{(p,m)}$, corresponding to the type decomposition $(\mathcal{P}_*V_P, g_P - \tilde{P}) = \bigoplus (\mathcal{P}_*V_{P,o}^{(p,m)}, g_{P,o}^{(p,m)})$ for any $P \in D$. We have the endomorphism $f = \sum_P \sum_{p,m,o} f_{P,o}^{(p,m)}$ of \mathcal{P}_*E , corresponding to g . Let $\mathcal{U} \subset \mathcal{P}_cE$ ($c \in \mathbb{R}$) be an $\mathcal{O}_{T \times \mathbb{P}^1}$ -submodule satisfying the following conditions:

- $\mathcal{U}_{|T \times \mathbb{C}} = \mathcal{P}_c E_{|T \times \mathbb{C}}$.
- $\mathcal{U}_{|T \times \{\infty\}}$ is semistable of degree 0.
- \mathcal{U} has a decomposition $\mathcal{U} = \bigoplus_P \bigoplus_{p,m,\mathbf{o}} \mathcal{U}_{P,\mathbf{o}}^{(p,m)}$ around $T \times \{\infty\}$, where $\mathcal{U}_{P,\mathbf{o}}^{(p,m)} := \mathcal{U} \cap \mathcal{P}_c E_{P,\mathbf{o}}^{(p,m)}$.
- $\mathcal{P}_{-1} E_{P,0}^{(p,m)} \subset \mathcal{U}_{P,0}^{(1,0)} \subset \mathcal{P}_0 E_{P,0}^{(p,m)}$ for any $P \in D$.

We define $N(\mathcal{U}) := R^1 p_{1*} (p_{12}^* \mathcal{Poin}^\vee \otimes p_{23}^* \mathcal{U})$.

Lemma 5.19 *We have $R^i p_{1*} (p_{12}^* \mathcal{Poin}^\vee \otimes p_{23}^* \mathcal{U}) = 0$ unless $i = 1$. In particular, $N(\mathcal{U})$ is a locally free \mathcal{O}_{T^\vee} -module.*

Proof The vanishing around $P \notin D$ is easy. We obtain the following object in $D^b(\mathcal{O}_{T^\vee \times \mathbb{P}^1})$:

$$\text{RFM}_-(\mathcal{U}) := p_{13*} (p_{12}^* \mathcal{Poin}^\vee \otimes p_{23}^* \mathcal{U})[1]$$

We can express $\text{RFM}_-(\mathcal{U})$ as a two term complex of locally free $\mathcal{O}_{T^\vee \times \mathbb{P}^1}$ modules $\mathcal{N}_{-1} \xrightarrow{a} \mathcal{N}_0$. Because a is generically isomorphism, it is injective. Hence, we have $\text{RFM}_-(\mathcal{U}) = \mathcal{H}^0(\text{RFM}_-(\mathcal{U}))$. We will not distinguish them.

Suppose $0 \in \mathcal{S}p_\infty(E)$. Let $U_0 \subset T^\vee$ denote a small neighbourhood of 0. Let $W_\infty \subset \mathbb{P}^1$ be a small neighbourhood of ∞ . We have the following decomposition:

$$\text{RFM}_-(\mathcal{U})|_{U_0 \times W_\infty} = \bigoplus_{p,m,\mathbf{o}} \text{RFM}_-(\mathcal{U}_{0,\mathbf{o}}^{(p,m)}).$$

If $(p, m, \mathbf{o}) \neq (1, 0, 0)$, the support of $\text{RFM}_-(\mathcal{U}_{0,\mathbf{o}}^{(p,m)})$ is proper over U_0 . Hence, we have the following decomposition:

$$\text{RFM}_-(\mathcal{U})|_{U_0 \times \mathbb{P}^1} = \bigoplus_{(p,m,\mathbf{o}) \neq (1,0,0)} \text{RFM}_-(\mathcal{U}_{0,\mathbf{o}}^{(p,m)}) \oplus \mathcal{M}(\mathcal{U}). \quad (59)$$

Here, $\mathcal{M}(\mathcal{U})|_{U_0 \times W_\infty} = \text{RFM}_-(\mathcal{U}_{0,0}^{(1,0)})$.

Let $\pi : T^\vee \times \mathbb{P}^1 \rightarrow T^\vee$ be the projection. By the condition **(A3)**, we have $R^1 \pi_* \text{RFM}_-(\mathcal{P}_{-1} E) = 0$. If $\mathcal{P}_{-1} E \subset \mathcal{U}$, we obtain $R^1 \pi_* \text{RFM}_-(\mathcal{U}) = 0$. It implies that $R^1 \pi_* \mathcal{M}(\mathcal{U}) = 0$ for such \mathcal{U} . By using the decomposition (59), we obtain $R^1 \pi_* \mathcal{M}(\mathcal{U}) = 0$ for any \mathcal{U} as above. Hence, we obtain the desired vanishing around $0 \in T^\vee$. We can use the same argument for the other points $P \in D$. \blacksquare

We have $N(\mathcal{U})(*D) = \text{Nahm}(\mathcal{P}_* E)$. By the proof, we obtain the following decomposition around any $P \in D$ induced by the decomposition (59):

$$N(\mathcal{U}) = \bigoplus_{p,m,\mathbf{o}} N(\mathcal{U}_{P,\mathbf{o}}^{(p,m)})$$

In particular, we have the following decomposition around any $P \in D$:

$$\text{Nahm}(\mathcal{P}_* E) = \bigoplus_{p,m,\mathbf{o}} \text{Nahm}(\mathcal{P}_* E)_{P,\mathbf{o}}^{(p,m)} \quad (60)$$

If $(p, m, \mathbf{o}) \neq (1, 0, 0)$, we have a natural isomorphism $\text{Nahm}(\mathcal{P}_* E)_{P,\mathbf{o}}^{(p,m)} \simeq \mathcal{N}^{(\infty,0)}(\mathcal{P}_* V_{P,\mathbf{o}}^{(p,m)}, g_{P,\mathbf{o}}^{(p,m)})$. Under the isomorphism, we define

$$\text{Nahm}_a(\mathcal{P}_* E)_{P,\mathbf{o}}^{(p,m)} := \mathcal{N}_a^{\infty,0}(\mathcal{P}_* V_{P,\mathbf{o}}^{(p,m)}, g_{P,\mathbf{o}}^{(p,m)}).$$

Let us consider the case $(p, m, \mathbf{o}) = (1, 0, 0)$. First, we define

$$\text{Nahm}_0(\mathcal{P}_* E)_{P,0}^{(1,0)} := N(\mathcal{P}_{-1} E)_{P,0}^{(1,0)}.$$

We set $\mathfrak{C}_P := \mathcal{P}_0 E_{P,0}^{(1,0)} / \mathcal{P}_{-1} E_{P,0}^{(1,0)}$. We have the following exact sequence around P :

$$0 \rightarrow N(\mathcal{P}_{-1} E)_{P,0}^{(1,0)} \rightarrow N(\mathcal{P}_0 E)_{P,0}^{(1,0)} \rightarrow R^1 p_{1*} (p_{12}^* \mathcal{Poin} \otimes p_{23}^* \mathfrak{C}_P) \rightarrow 0$$

The multiplication of ζ on $R^1 p_{1*}(p_{12}^* \mathcal{Poin} \otimes p_{23}^* \mathfrak{C}_P)$ is 0 by construction. Hence, we have the induced surjection:

$$N(\mathcal{P}_0 E)_{P,0|0}^{(1,0)} := N(\mathcal{P}_0 E)_{P,0}^{(1,0)} \otimes \mathcal{O}_P \longrightarrow R^1 p_{1*}(p_{12}^* \mathcal{Poin} \otimes p_{23}^* \mathfrak{C}_P)$$

Let K denote the kernel. We have the following morphisms:

$$R^1 p_{1*}(p_{12}^* \mathcal{Poin} \otimes p_{23}^* \mathfrak{C}_P) \simeq N(\mathcal{P}_0 E)_{P,0|0}^{(1,0)} / K \xrightarrow{h} N(\mathcal{P}_{-1} E)_{P,0|0}^{(1,0)}$$

Here, h is the injection induced by the multiplication of ζ . We have a natural isomorphism of \mathbb{C} -vector spaces:

$$R^1 p_{1*}(p_{12}^* \mathcal{Poin} \otimes p_{23}^* \mathfrak{C}_P) \simeq \mathcal{P}_0 V_{P,0|\infty}^{(1,0)}$$

Hence, for any $-1 < c < 0$, we define

$$F_c(\mathrm{Nahm}_0(\mathcal{P}_* E)_{P,0}^{(1,0)} \otimes \mathcal{O}_P) := F_c(\mathcal{P}_0 V_{P,0|\infty}^{1,0}).$$

We also set $F_0(\mathrm{Nahm}_0(\mathcal{P}_* E)_{P,0}^{(1,0)} \otimes \mathcal{O}_P) = \mathrm{Nahm}_0(\mathcal{P}_* E)_{P,0}^{(1,0)} \otimes \mathcal{O}_P$. The filtration of $\mathrm{Nahm}_0(\mathcal{P}_* E)_{P,0}^{(1,0)} \otimes \mathcal{O}_P$ indexed by $] -1, 0]$ induces a filtered bundle $\mathrm{Nahm}_*(\mathcal{P}_* E)_{P,0}^{(1,0)}$ over $\mathrm{Nahm}(\mathcal{P}_* E)_{P,0}^{(1,0)}$. In all, we obtain a filtered bundle $\mathrm{Nahm}_*(\mathcal{P}_* E)$ over $\mathrm{Nahm}(\mathcal{P}_* E)$. We also denote the filtered Higgs bundle $(\mathrm{Nahm}_*(\mathcal{P}_* E), \theta)$ just by $\mathrm{Nahm}_*(\mathcal{P}_* E)$.

Remark 5.20 We obtain a different transformation by replacing \mathcal{Poin} with \mathcal{Poin}^\vee , for which we can argue in a similar way. \blacksquare

5.3.3 Inversion

Proposition 5.21

- Let $(\mathcal{P}_* \mathcal{E}, \theta)$ be an admissible filtered Higgs bundle on (T^\vee, D) satisfying **(A0)**. Then, $\mathrm{Nahm}_*(\mathcal{P}_* \mathcal{E}, \theta)$ satisfies the conditions **(A1–3)**, and we have a natural isomorphism $\mathrm{Nahm}_*(\mathrm{Nahm}_*(\mathcal{P}_* \mathcal{E}, \theta)) \simeq (\mathcal{P}_* \mathcal{E}, \theta)$.
- Let $\mathcal{P}_* E$ be a filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$ satisfying the conditions **(A1–3)**. Then, $\mathrm{Nahm}_*(\mathcal{P}_* E)$ is admissible and satisfies the condition **(A0)**. We have a natural isomorphism $\mathrm{Nahm}_*(\mathrm{Nahm}_*(\mathcal{P}_* E)) \simeq \mathcal{P}_* E$.

Proof For $I \subset \{1, 2, 3\}$, let p_I be the projection of $T^\vee \times T \times \mathbb{P}^1$ onto the product of the i -th components ($i \in I$). Let $(\mathcal{P}_* E, \theta)$ be an admissible Higgs bundle on (T^\vee, D) . Clearly, $\mathrm{Nahm}(\mathcal{P}_* E, \theta)$ satisfies **(A1)**. It satisfies **(A2)** by Proposition 5.7. Let $L \in T^\vee$. Let us consider the following complex on $T^\vee \times T \times \mathbb{P}^1$:

$$\tilde{\mathcal{C}}^0 = p_{1*} \mathcal{C}^0(\mathcal{P}_* \mathcal{E}, \theta) \otimes p_{23}^* \mathcal{Poin} \otimes p_2^* L^\vee \otimes p_3^* \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\theta + wd\zeta} \tilde{\mathcal{C}}^1 = p_{1*} \mathcal{C}^1(\mathcal{P}_* \mathcal{E}, \theta) \otimes p_{23}^* \mathcal{Poin} \otimes p_2^* L^\vee.$$

We have $p_{12*} \tilde{\mathcal{C}}^\bullet \simeq R^1 p_{12*} \tilde{\mathcal{C}}^\bullet[-1] \simeq \mathcal{C}^1(\mathcal{P}_* \mathcal{E}, \theta) \otimes \mathcal{Poin} \otimes L^\vee[-1]$ on $T^\vee \times T$. For the projection $\pi : T^\vee \times T \longrightarrow T^\vee$, we have $\pi_*(\mathcal{C}^1(\mathcal{P}_* \mathcal{E}, \theta) \otimes \mathcal{Poin} \otimes L^\vee) \simeq R^1 \pi_*(\mathcal{C}^1(\mathcal{P}_* \mathcal{E}, \theta) \otimes \mathcal{Poin} \otimes L^\vee)[-1]$, which is a skyscraper sheaf $\mathcal{C}^1(\mathcal{P}_* \mathcal{E}, \theta)|_L$ at L . Hence, we have

$$\mathbb{H}^i(T^\vee \times T \times \mathbb{P}^1, \tilde{\mathcal{C}}^\bullet) = \begin{cases} 0 & (i \neq 2) \\ \mathcal{C}^1(\mathcal{P}_* \mathcal{E}, \theta)|_L & (i = 2) \end{cases}$$

We can deduce $H^i(T \times \mathbb{P}^1, \mathbf{N}(\mathcal{P}_* \mathcal{E}, \theta)) = 0$ unless $i = 1$. We have

$$p_{12*}(\tilde{\mathcal{C}}^\bullet \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq R^1 p_{12*}(\tilde{\mathcal{C}}^\bullet \otimes \mathcal{O}_{\mathbb{P}^1}(-1))[-1] \simeq \mathcal{C}^0(\mathcal{P}_* \mathcal{E}, \theta) \otimes \mathcal{Poin} \otimes L^\vee[-1]$$

on $T \times T^\vee$. Hence, we have

$$\mathbb{H}^i(T^\vee \times T \times \mathbb{P}^1, \tilde{\mathcal{C}}^\bullet \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) = \begin{cases} 0 & (i \neq 2) \\ \mathcal{C}^0(\mathcal{P}_* \mathcal{E}, \theta)|_L & (i = 2) \end{cases}$$

We can deduce $H^i(T \times \mathbb{P}^1, \mathbf{N}(\mathcal{P}_*\mathcal{E}, \theta) \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. By the argument in Lemma 5.19, we obtain that $\text{Nahm}(\mathcal{P}_*\mathcal{E}, \theta)$ satisfies the condition **(A3)**. By the same argument, we obtain that

$$\text{Nahm}(\text{Nahm}(\mathcal{P}_*\mathcal{E}, \theta)) \otimes \mathcal{O}(*D) \simeq (\mathcal{E}, \theta).$$

If $(p, m, \mathbf{o}) \neq (1, 0, 0)$, we obtain the comparison of the filtered bundles over $\mathcal{E}_{P, \mathbf{o}}^{(p, m)}$ from Proposition 5.7. We obtain the comparison of the filtered bundles over $\mathcal{E}_{P, 0}^{(1, 0)}$ directly from the construction. Thus, we obtain the first claim.

Let \mathcal{P}_*E be the filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$ satisfying **(A1–3)**. By using the description in §5.3.2 with Lemma 5.5, we obtain that $\text{Nahm}(\mathcal{P}_*E)$ is admissible. Let $\mathcal{V} \subset \mathcal{P}_0E$ be determined by the conditions (i) $\mathcal{V} = \mathcal{P}_0E$ outside of D , (ii) we have a decomposition $\mathcal{V} = \bigoplus \mathcal{V}_{P, \mathbf{o}}^{(p, m)}$ around each $P \in D$ such that

$$\mathcal{V}_{P, \mathbf{o}}^{(p, m)} = \begin{cases} \mathcal{P}_0E_{P, 0}^{(1, 0)}, & ((p, m, \mathbf{o}) = (1, 0, 0)), \\ \mathcal{P}_{-1/2}E_{P, \mathbf{o}}^{(p, m)}, & (\text{otherwise}). \end{cases}$$

We have $\mathcal{C}^0(\text{Nahm}(\mathcal{P}_*E)) = N(\mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$ and $\mathcal{C}^1(\text{Nahm}(\mathcal{P}_*E)) = N(\mathcal{V})$. The differential is induced by the multiplication of $-w$. We shall rewrite

$$\mathcal{C}^0(\text{Nahm}(\mathcal{P}_*E)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\theta + wd\zeta} \mathcal{C}^1(\text{Nahm}(\mathcal{P}_*E)). \quad (61)$$

For $I \subset \{1, 2, 3, 4, 5\}$, let p_I denote the projection of $T \times \mathbb{P}^1 \times T^\vee \times T \times \mathbb{P}^1$ onto the product of the i -th components ($i \in I$). We set

$$C_0 := p_{12}^*(\mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) \otimes p_{13}^*\mathcal{Poin}^\vee \otimes p_{34}^*\mathcal{Poin} \otimes p_5^*\mathcal{O}_{\mathbb{P}^1}(-1)$$

$$C_1 := p_{12}^*\mathcal{V} \otimes p_{13}^*\mathcal{Poin}^\vee \otimes p_{34}^*\mathcal{Poin}$$

We regard $\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_{\mathbb{P}^1}(\{\infty\})$, and let $\iota : \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{O}_{\mathbb{P}^1}(\{\infty\})$ be the natural inclusion. Let $G : C_0 \rightarrow C_1$ be induced by $-p_2^*w \otimes p_5^*\iota + p_2^*\iota \otimes p_5^*w$. Then, (61) is naturally isomorphic to $R^1p_{345*}(C_0 \xrightarrow{G} C_1)$.

For $I \subset \{1, 2, 3, 4\}$ let q_I denote the projection of $T \times T^\vee \times T \times \mathbb{P}^1$ onto the product of the i -th components ($i \in I$). The complex $p_{1345*}(C_0 \xrightarrow{G} C_1)$ is quasi isomorphic to

$$q_{14}^*\mathcal{V} \otimes q_{12}^*\mathcal{Poin}^\vee \otimes q_{23}^*\mathcal{Poin}[-1].$$

For $I \subset \{1, 2, 3\}$, let s_I denote the projection of $T \times T \times \mathbb{P}^1$ onto the product of the i -th components ($i \in I$). We have the following natural isomorphism

$$q_{134*}(q_{14}^*\mathcal{V} \otimes q_{12}^*\mathcal{Poin}^\vee \otimes q_{23}^*\mathcal{Poin}[-1]) \simeq s_{13}^*\mathcal{V} \otimes s_{12}^*\mathcal{O}_\Delta[-2]$$

Here, \mathcal{O}_Δ denote the structure sheaf of the diagonal in $T \times T$. Then, we obtain a natural isomorphism

$$\mathcal{V} \simeq \mathbf{N}(\text{Nahm}(\mathcal{P}_*E, \theta))$$

as $\mathcal{O}_{T \times \mathbb{P}^1}$ -modules. In particular, we obtain that **(A0)** is satisfied for $\text{Nahm}(\mathcal{P}_*E, \theta)$. If $(p, m, \mathbf{o}) \neq (1, 0, 0)$, we obtain the comparison of the filtered bundles over $\mathcal{V}(* (T \times \{\infty\}))_{P, \mathbf{o}}^{(p, m)}$ from Proposition 5.7. The comparison in the case $(p, m, \mathbf{o}) = (1, 0, 0)$ follows directly from the construction. \blacksquare

Corollary 5.22 *Let \mathcal{P}_*E be a filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$ satisfying the conditions **(A1–3)**. We have*

$$\deg(\mathcal{P}_*E) = \deg(\text{Nahm}(\mathcal{P}_*E))$$

Proof It follows from Proposition 5.15 and Proposition 5.21. \blacksquare

5.4 Refinement for good filtered Higgs bundles

5.4.1 Good filtered Higgs bundles

Let X and D be as in §5.1.1. Let (\mathcal{P}_*V, θ) be a good filtered Higgs bundle on (X, D) . Namely, there exists a ramified covering $\varphi_p : (X^{(p)}, D^{(p)}) \rightarrow (X, D)$ given by $\varphi_p(z_p) = z_p^p$ with a decomposition

$$\varphi_p^*(\mathcal{P}_*V, \theta) = \bigoplus_{\mathfrak{a} \in z_p^{-1}\mathbb{C}[z_p^{-1}]} (\mathcal{P}_*V_{\mathfrak{a}}^{(p)}, \theta_{\mathfrak{a}}^{(p)}), \quad (62)$$

such that $\theta_{\mathfrak{a}}^{(p)} - d\mathfrak{a} \text{id}_{V_{\mathfrak{a}}^{(p)}}$ is logarithmic. Let $\text{Irr}(\varphi_p^*\theta)$ denote the set of \mathfrak{a} such that $V_{\mathfrak{a}}^{(p)} \neq 0$. The Galois group $\text{Gal}(\varphi_p)$ naturally acts on $\varphi_p^*(\mathcal{P}_*V, \theta)$ and $\text{Irr}(\varphi_p^*\theta)$. The quotient set $\text{Irr}(\varphi_p^*\theta)/\text{Gal}(\varphi_p)$ is denoted by $\mathbf{Irr}(\varphi_p^*\theta)$. We have the orbit decomposition $\text{Irr}(\varphi_p^*\theta) = \coprod_{\mathfrak{o} \in \mathbf{Irr}(\varphi_p^*\theta)} \mathfrak{o}$. We set $(\mathcal{P}_*V_{\mathfrak{o}}^{(p)}, \theta_{\mathfrak{o}}^{(p)}) := \bigoplus_{\mathfrak{a} \in \mathfrak{o}} (\mathcal{P}_*V_{\mathfrak{a}}^{(p)}, \theta_{\mathfrak{a}}^{(p)})$. We obtain a $\text{Gal}(\varphi_p)$ -equivariant decomposition $\varphi_p^*(\mathcal{P}_*V, \theta) = \bigoplus_{\mathfrak{o} \in \mathbf{Irr}(\varphi_p^*\theta)} (\mathcal{P}_*V_{\mathfrak{o}}^{(p)}, \theta_{\mathfrak{o}}^{(p)})$. By the descent, we obtain a decomposition

$$(\mathcal{P}_*V, \theta) = \bigoplus_{\mathfrak{o} \in \mathbf{Irr}(\varphi_p^*\theta)} (\mathcal{P}_*V_{\mathfrak{o}}, \theta_{\mathfrak{o}}). \quad (63)$$

If we have a factorization $\varphi_p = \varphi_{p_1} \circ \varphi_{p_2}$ such that $\varphi_{p_2}^*(\mathcal{P}_*V, \theta)$ has a decomposition as above. Then, φ_{p_1} gives a bijection $\text{Irr}(\varphi_{p_2}^*\theta) \simeq \text{Irr}(\varphi_p^*\theta)$. It induces a bijection of the quotient sets by the Galois groups. By the identification, we denote them by $\text{Irr}(\theta)$ and $\mathbf{Irr}(\theta)$. The decomposition (63) is independent of the choice of φ_p .

For each $\mathfrak{o} \in \mathbf{Irr}(\theta)$, there exists a minimum $p_{\mathfrak{o}}$ among the numbers p such that $\varphi_p^*(\mathcal{P}_*V_{\mathfrak{o}}, \theta_{\mathfrak{o}})$ has a decomposition such as (62). In this case, we have $|\mathfrak{o}| = p_{\mathfrak{o}}$. We set $X^{\mathfrak{o}} := X^{(p_{\mathfrak{o}})}$, $\varphi_{\mathfrak{o}} := \varphi_{p_{\mathfrak{o}}}$ and $z_{\mathfrak{o}} := z_{p_{\mathfrak{o}}}$. We have the decomposition on $X^{\mathfrak{o}}$:

$$\varphi_{\mathfrak{o}}^*(\mathcal{P}_*V_{\mathfrak{o}}, \theta_{\mathfrak{o}}) = \bigoplus_{\mathfrak{a} \in \mathfrak{o}} (\mathcal{P}_*V_{\mathfrak{a}}^{\mathfrak{o}}, \theta_{\mathfrak{a}}^{\mathfrak{o}}). \quad (64)$$

For any $\mathfrak{a} \in \mathfrak{o}$, we have a natural isomorphism $(\mathcal{P}_*V_{\mathfrak{o}}, \theta_{\mathfrak{o}}) \simeq \varphi_{\mathfrak{o}*}(\mathcal{P}_*V_{\mathfrak{a}}^{\mathfrak{o}}, \theta_{\mathfrak{a}}^{\mathfrak{o}})$. We set $m_{\mathfrak{o}} := (\text{ord}_{z_{\mathfrak{o}}^{-1}} \mathfrak{a})$ which is independent of $\mathfrak{a} \in \mathfrak{o}$.

If $(\mathcal{P}_*V, \theta) = (\mathcal{P}_*V_{\mathfrak{o}}, \theta_{\mathfrak{o}})$, we say that (\mathcal{P}_*V, θ) has pure irregularity \mathfrak{o} in this paper.

If X is shrank appropriately, we have the following decomposition, which is a refinement of (62),

$$\varphi_p^*(\mathcal{P}_*V, \theta) = \bigoplus_{\mathfrak{a} \in z_p^{-1}\mathbb{C}[z_p^{-1}]} \bigoplus_{\alpha \in \mathbb{C}} (\mathcal{P}_*V_{\mathfrak{a}, \alpha}^{(p)}, \theta_{\mathfrak{a}, \alpha}^{(p)})$$

such that the eigenvalues of the residues $\text{Res}(\theta_{\mathfrak{a}, \alpha}^{(p)} - (d\mathfrak{a} + p\alpha dz/z) \text{id}_{V_{\mathfrak{a}, \alpha}^{(p)}})$ are 0. Let $(\mathcal{P}_*V_{\mathfrak{o}, \alpha}, \theta_{\mathfrak{o}, \alpha})$ be the descent of $\bigoplus_{\mathfrak{a} \in \mathfrak{o}} (\mathcal{P}_*V_{\mathfrak{a}, \alpha}^{(p)}, \theta_{\mathfrak{a}, \alpha}^{(p)})$ to X . We obtain a decomposition

$$(\mathcal{P}_*V, \theta) = \bigoplus_{\mathfrak{o} \in \mathbf{Irr}(\theta)} \bigoplus_{\alpha \in \mathbb{C}} (\mathcal{P}_*V_{\mathfrak{o}, \alpha}, \theta_{\mathfrak{o}, \alpha}).$$

On $X^{\mathfrak{o}}$, we have a decomposition:

$$\varphi_{\mathfrak{o}}^*(\mathcal{P}_*V_{\mathfrak{o}, \alpha}, \theta_{\mathfrak{o}, \alpha}) = \bigoplus_{\mathfrak{a} \in \mathfrak{o}} (\mathcal{P}_*V_{\mathfrak{a}, \alpha}^{\mathfrak{o}}, \theta_{\mathfrak{a}, \alpha}^{\mathfrak{o}})$$

Lemma 5.23 *Let (\mathcal{P}_*V, θ) be a good filtered Higgs bundle. Let \mathcal{P}_*V' be a strict Higgs subbundle, i.e., \mathcal{P}_*V' is a strict filtered subbundle of \mathcal{P}_*V such that $\theta(V') \subset V' \otimes \Omega^1$. The restriction of θ to V' is denoted by θ' . Then, $(\mathcal{P}_*V', \theta')$ is also good.*

Proof Suppose (\mathcal{P}_*V, θ) is unramified with the decomposition $(\mathcal{P}_*V, \theta) = \bigoplus (\mathcal{P}_*V_{\mathfrak{a}}, \theta_{\mathfrak{a}})$. Because $\theta(V') \subset V' \otimes \Omega_X^1$, we have $V' = \bigoplus (V' \cap V_{\mathfrak{a}})$. By the strictness, we obtain $\mathcal{P}_*V' = \bigoplus (V' \cap \mathcal{P}_*V_{\mathfrak{a}})$. Hence, $(\mathcal{P}_*V', \theta')$ is good. The ramified case can be reduced to the unramified case by the descent. \blacksquare

Take $p \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}_{\geq 0}$ with $\text{g.c.d.}(p, m) = 1$. Let $\mathbf{Irr}(\theta, p, m) := \{\mathfrak{o} \in \mathbf{Irr}(\theta) \mid p_{\mathfrak{o}}/m_{\mathfrak{o}} = p/m\}$. We have

$$\mathcal{P}_* V^{(p,m)} = \bigoplus_{\mathfrak{o} \in \mathbf{Irr}(\theta, p, m)} \mathcal{P}_* V_{\mathfrak{o}}$$

For any $\mathfrak{o} \in \mathcal{J}(p, m)$, we have $\mathbf{Irr}(\theta, p, m, \mathfrak{o}) \subset \mathbf{Irr}(\theta, p, m)$ such that

$$\mathcal{P}_* V_{\mathfrak{o}}^{(p,m)} = \bigoplus_{\mathfrak{o} \in \mathbf{Irr}(\theta, p, m, \mathfrak{o})} \mathcal{P}_* V_{\mathfrak{o}}.$$

Take any $\alpha \in \mathfrak{o}$. For each $\mathfrak{o} \in \mathbf{Irr}(\theta, p, m, \mathfrak{o})$, we have $\mathfrak{a} \in \mathfrak{o}$ such that

$$\mathcal{P}_* V_{\alpha}^{(p)} = \bigoplus_{\mathfrak{o} \in \mathbf{Irr}(\theta, p, m, \mathfrak{o})} \varphi_{(p_{\mathfrak{o}}/p)} * (\mathcal{P}_* V_{\mathfrak{a}}^{\mathfrak{o}}).$$

Here, $\varphi_{p_{\mathfrak{o}}/p}$ is the ramified covering $X^{\mathfrak{o}} \rightarrow X^{(p)}$ given by $\varphi_{p_{\mathfrak{o}}/p}(z_{\mathfrak{o}}) = z_{\mathfrak{o}}^{p_{\mathfrak{o}}/p}$. Let $c \in \mathbb{R}$. We take a frame $\mathbf{v}_{\mathfrak{o}} = (v_{\mathfrak{o},i})$ of $\mathcal{P}_{p_{\mathfrak{o}}c} V_{\mathfrak{a}}^{\mathfrak{o}}$ compatible with the parabolic structure. Then, the tuple of the sections

$$\left\{ z_{\mathfrak{o}}^j v_{\mathfrak{o},i} \mid \mathfrak{o} \in \mathbf{Irr}(\theta, p, m, \mathfrak{o}), \ 1 \leq i \leq \text{rank } V_{\mathfrak{a}}^{\mathfrak{o}}, \ 0 \leq j < p_{\mathfrak{o}}/p \right\}$$

gives a frame of $\mathcal{P}_{pc} V_{\alpha}^{(p)}$.

5.4.2 Description of the parabolic structure of $\mathcal{N}_*^{0,\infty}(\mathcal{P}_* V, \theta)$

Let U_{ζ} be a small neighborhood of 0 in \mathbb{C}_{ζ} . Let $(\mathcal{P}_* V, \theta)$ be a good filtered Higgs bundle on $(U_{\zeta}, 0)$. For simplicity, we assume that $(\mathcal{P}_* V, \theta)$ has slope (p, m) . We take $\mathfrak{a} \in \mathfrak{o}$ for each $\mathfrak{o} \in \mathbf{Irr}(\theta)$. Let $c \in \mathbb{R}$. We take a frame $\mathbf{v}_{\mathfrak{o}} = (v_{\mathfrak{o},i})$ of $\mathcal{P}_{cp_{\mathfrak{o}}} V_{\mathfrak{a}}^{\mathfrak{o}}$ compatible with the parabolic structure. Each $\zeta_{\mathfrak{o}}^j v_{\mathfrak{o},i} dz_{\mathfrak{o}}/z_{\mathfrak{o}}$ induces a section of $\mathcal{N}^{\infty,0}(\mathcal{P}_* V, \theta)$, denoted by $[\zeta_{\mathfrak{o}}^j v_{\mathfrak{o},i} dz_{\mathfrak{o}}/z_{\mathfrak{o}}]$. The following lemma is clear by the construction of the filtered bundle $\mathcal{N}_*^{0,\infty}(\mathcal{P}_* V, \theta)$. (See the proof of Lemma 5.3.)

Lemma 5.24 *The tuple*

$$\left\{ [\zeta_{\mathfrak{o}}^j v_{\mathfrak{o},i} d\zeta_{\mathfrak{o}}/\zeta_{\mathfrak{o}}] \mid \mathfrak{o} \in \mathbf{Irr}(\theta), \ 0 \leq j < p_{\mathfrak{o}} + m_{\mathfrak{o}}, \ 1 \leq i \leq \text{rank } V_{\mathfrak{a}}^{\mathfrak{o}} \right\}$$

is a frame of $\mathcal{N}_{\kappa_1(p,m,c)}^{0,\infty}(\mathcal{P}_* V, \theta)$, compatible with the parabolic structure. If the parabolic degree of $v_{\mathfrak{o},i}$ is b , the parabolic degree of $[\zeta_{\mathfrak{o}}^j v_{\mathfrak{o},i} d\zeta_{\mathfrak{o}}/\zeta_{\mathfrak{o}}]$ is $(b - j - m_{\mathfrak{o}}/2)(p_{\mathfrak{o}} + m_{\mathfrak{o}})^{-1}$. \blacksquare

5.4.3 Description of the parabolic structure of $\mathcal{N}_*^{\infty,0}(\mathcal{P}_* V, g)$

Let $(\mathcal{P}_* V, g)$ be a filtered bundle with an endomorphism on $(U_{\tau}, 0)$ such that $\mathcal{P}_* V$ with $\psi := -\tau^{-2} g d\tau$ is good filtered Higgs bundle. For simplicity, we assume that $(\mathcal{P}_* V, g)$ has a slope (p, m) with $p > m \neq 0$.

We take $\mathfrak{a} \in \mathfrak{o}$ for each $\mathfrak{o} \in \mathbf{Irr}(\psi)$. Let $c \in \mathbb{R}$. We take a frame $\mathbf{v}_{\mathfrak{o}} = (v_{\mathfrak{o},i})$ of $\mathcal{P}_{cp_{\mathfrak{o}}} V_{\mathfrak{a}}^{\mathfrak{o}}$ compatible with the parabolic structure. Each $\tau_{\mathfrak{o}}^j v_{\mathfrak{o},i}$ induces a section of $\mathcal{N}^{\infty,0}(\mathcal{P}_* V, g)$, denoted by $[\tau_{\mathfrak{o}}^j v_{\mathfrak{o},i}]$. The following lemma is clear by the construction of the filtered bundle $\mathcal{N}_*^{\infty,0}(\mathcal{P}_* V, g)$. (See the proof of Lemma 5.5.)

Lemma 5.25 *The tuple*

$$\left\{ [\tau_{\mathfrak{o}}^j v_{\mathfrak{o},i}] \mid \mathfrak{o} \in \mathbf{Irr}(\psi), \ 0 \leq j < p_{\mathfrak{o}} - m_{\mathfrak{o}}, \ 1 \leq i \leq \text{rank } V_{\mathfrak{a}}^{\mathfrak{o}} \right\}$$

is a frame of $\mathcal{N}_{\kappa_2(p,m,c)}^{\infty,0}(\mathcal{P}_* V, g)$ compatible with the parabolic structure. If the parabolic degree of $v_{\mathfrak{o},i}$ is b , the parabolic degree of $\tau_{\mathfrak{o}}^j v_{\mathfrak{o},i}$ is $(b - j + p_{\mathfrak{o}} - m_{\mathfrak{o}}/2)(p_{\mathfrak{o}} - m_{\mathfrak{o}})^{-1}$. \blacksquare

5.4.4 A stationary phase formula

We have the following type of stationary phase formula for the local Nahm transform, which is analogue of the stationary phase formula for the local Fourier transform (See [16], [19], [22], [33], [35], and [43].)

Theorem 5.26 *Let (\mathcal{P}_*V, θ) be an admissible filtered Higgs bundle on U_ζ .*

- (\mathcal{P}_*V, θ) is good, if and only if $\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta)$ is good.
- Suppose $(\mathcal{P}_*V, \theta) \simeq \varphi_{p*}(\mathcal{P}_*V', \theta')$, where $\theta' - d\mathbf{a}$ is logarithmic for some $\mathbf{a} \in \zeta_p^{-1}\mathbb{C}[\zeta_p^{-1}]$ with $\deg_{\zeta_p^{-1}} \mathbf{a} = m > 0$. Then, there exists (\mathcal{P}_*W', ψ') on $U_{\tau_{p+m}}$ such that (i) $\psi' - d\mathbf{b}$ is logarithmic for some $\mathbf{b} \in \tau_{p+m}^{-1}\mathbb{C}[\tau_{p+m}^{-1}]$ with $\deg_{\tau_{p+m}^{-1}} \mathbf{b} = m$, (ii) we have an isomorphism

$$\varphi_{p+m*}(\mathcal{P}_*W', \psi') \simeq \mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta).$$

Moreover, we have an isomorphism $\mathrm{Gr}_c^{\mathcal{P}}(V') \simeq \mathrm{Gr}_{c-m/2}^{\mathcal{P}}(W')$ under which $\mathrm{Res}(\varphi_p^*\theta) = \mathrm{Res}(\varphi_{p+m}^*\psi')$. (The choice of \mathbf{b} will be explained in the proof.)

- If (\mathcal{P}_*V, θ) is logarithmic, $(\mathcal{P}_*W, \psi) := \mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta)$ is also logarithmic. Moreover, we have an isomorphism

$$\mathrm{Gr}_c^{\mathcal{P}}(W) \simeq \begin{cases} \mathrm{Gr}_c^{\mathcal{P}}(V) & (-1 < c < 0) \\ \mathrm{Im}(\mathrm{Res}(\theta) : \mathrm{Gr}_0^{\mathcal{P}}(V) \longrightarrow \mathrm{Gr}_0^{\mathcal{P}}(V)) & (c = 0) \end{cases}$$

Under the isomorphism, we have $\mathrm{Res}(\psi) = \mathrm{Res}(\theta)$.

Proof Let us begin with the third claim. We obtain the isomorphism of the associated graded vector spaces by the construction of \mathcal{P}_*W . We have the expression $\theta = f d\zeta/\zeta$, where f is an endomorphism of \mathcal{P}_*V . It naturally induces an endomorphism f' of \mathcal{P}_*W , and we have $\psi = f' d\tau/\tau$ by the construction. Thus, we obtain the third claim.

Let us consider the second claim. Our argument is close to that in [16]. To simplify the notation, we set $\eta := \tau_{p+m}$ and $u := \zeta_p$. We set $G(u) := u\partial_u \mathbf{a}(u) = \sum_{j=1}^m \alpha_j u^{-j}$. Let $\omega := e^{2\pi\sqrt{-1}/(p+m)}$. We have holomorphic functions $u^{(i)}(\eta)$ ($i = 0, \dots, p+m-1$) on U_η , satisfying $\partial_\eta u^{(i)}(0) = \partial_\eta u^{(0)}(0)\omega^i$ and

$$G(u^{(i)}(\eta)) + pu^{(i)}(\eta)^p/\eta^{p+m} = 0.$$

For any $c \in \mathbb{R}$, we consider $\mathcal{P}_{c-m/2}\mathcal{V} := \mathcal{P}_{c-m/2}\varphi_{p+m}^*\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta)$. We take a frame \mathbf{v} of \mathcal{P}_cV' compatible with the parabolic structure. We put $\tilde{\nu}_{ij} := (\eta^{-1}u)^i v_j$ ($0 \leq i \leq p+m-1$, $1 \leq j \leq \mathrm{rank} V'$). They induce a frame of $\mathcal{P}_{c-m/2}\mathcal{V}$, which is compatible with the parabolic structure. By the frames, for $c-1 < d \leq c$, we obtain an isomorphism

$$\mathrm{Gr}_{d-m/2}^{\mathcal{P}}(\mathcal{V}) \simeq \mathrm{Gr}_d^{\mathcal{P}}(V') \otimes \mathbb{C}^{p+m}. \quad (65)$$

The following lemma can be checked by a direct computation.

Lemma 5.27 $\eta^{-1}u$ gives an endomorphism F of $\mathcal{P}_*\mathcal{V}$. On $\mathrm{Gr}_{d-m/2}^{\mathcal{P}}(\mathcal{V})$, we have

$$F(\tilde{\nu}_{i,j}) = \begin{cases} \tilde{\nu}_{i+1,j} & (i < p+m-1) \\ -p^{-1}\alpha_m \tilde{\nu}_{0,j} & (i = p+m-1) \end{cases}$$

The eigenvalues of F on $\mathrm{Gr}^{\mathcal{P}}$ are $\partial_\eta u^{(i)}(0)$ ($i = 0, \dots, p+m-1$). ■

By the lemma, we obtain the decomposition $(\mathcal{P}_*\mathcal{V}, F) = \bigoplus_{j=0}^{p+m-1} (\mathcal{P}_*\mathcal{V}^{(j)}, F^{(j)})$ such that $F_{|0}^{(j)}$ has a unique eigenvalue $\partial_\eta u^{(j)}(0)$. Note that $\mathcal{N}_*^{0,\infty}(\mathcal{P}_*V, \theta) \simeq \varphi_{p+m*}(\mathcal{P}_*\mathcal{V}^{(0)}, \zeta(-\tau^{-2}d\tau))$. We also have an isomorphism $\mathrm{Gr}_c^{\mathcal{P}}(V') \simeq \mathrm{Gr}_{c-m/2}^{\mathcal{P}}(\mathcal{V}^{(0)})$.

We have the expression $\theta_{\mathbf{a}} = (G(u) + f) du/u$, where f is an endomorphism of \mathcal{P}_*V' . On $\mathcal{P}_{c-m/2}\mathcal{V}$, we have $\eta^m(G(u) + pu^p/\eta^{p+m}) = -\eta^m f$. We have the following decomposition:

$$\eta^m(G(u) + pu^p/\eta^{p+m}) = (\eta^{-1}u - \eta^{-1}u^{(0)}(\eta)) \times p \prod_{i=1}^{p+m-1} (\eta^{-1}u - \eta^{-1}u^{(i)}(\eta)) (\eta^{-1}u)^{-m}$$

Because $\eta^{-1}u - \eta^{-1}u^{(j)}(\eta)$ ($1 \leq j < p-m$) are invertible on $\mathcal{P}_{c-m/2}\mathcal{V}^{(0)}$, we obtain the following on $\mathcal{P}_{c-m/2}\mathcal{V}^{(0)}$:

$$\eta^{-1}u - \eta^{-1}u^{(0)}(\eta) = -p^{-1}\eta^m \cdot f \cdot \prod_{j=1}^{p+m-1} (\eta^{-1}u - \eta^{-1}u^{(j)}(\eta))^{-1} (\eta^{-1}u)^m$$

Let $Q_k(x, y) = \sum_{i+j=k} x^i y^j$. We have

$$\begin{aligned} \zeta/\tau - \frac{u^{(0)}(\eta)^p}{\eta^{p+m}} &= \eta^{-m} (\eta^{-1}u - \eta^{-1}u^{(0)}(\eta)) \cdot Q_{p-1}(\eta^{-1}u, \eta^{-1}u^{(0)}(\eta)) \\ &= -f \prod_{j=1}^{p+m-1} (\eta^{-1}u - \eta^{-1}u^{(j)}(\eta)) (\eta^{-1}u)^m p^{-1} Q_{p-1}(\eta^{-1}u, \eta^{-1}u^{(0)}(\eta)) \end{aligned} \quad (66)$$

Hence, we obtain that $(\zeta/\tau - u^{(0)}(\eta)^p \eta^{-p-m}) \mathcal{P}_* \mathcal{V}^{(0)} \subset \mathcal{P}_* \mathcal{V}^{(0)}$. Moreover, on $\text{Gr}_a^{\mathcal{P}}(\mathcal{V}^{(0)})$, the endomorphisms u/η and $u^{(0)}(\eta)/\eta$ are the multiplication of $\partial_{\eta} u^{(0)}(0)$. Hence, $(\zeta/\tau - u^{(0)}(\eta)^p \eta^{-p-m})$ acts as $-(p+m)^{-1}f$ on $\text{Gr}^{\mathcal{P}}(\mathcal{V})$. We set $\mathcal{P}_*W' := \mathcal{P}_* \mathcal{V}^{(0)}$ and $\psi' := -\zeta \tau^{-2} d\tau = -(\zeta/\tau)(p+m)d\eta/\eta$. We have $\mathbf{b} \in \eta^{-1}\mathbb{C}[\eta^{-1}]$ uniquely determined by the condition that $\eta \partial_{\eta} \mathbf{b}$ is equal to the polar part of $-(p+m)u^{(0)}(\eta)^p \eta^{-p-m}$. Then, $\psi' - d\mathbf{b}$ is logarithmic. The residue acts as f . Hence, the second claim of Theorem 5.26 follows. It also implies the “only if” part in the first claim.

Let us prove the “if” part of the first claim. We use the inverse transform. Let (\mathcal{P}_*W, ψ) be a good filtered Higgs bundle on $(U_{\tau}, 0)$ which is isomorphic to $\varphi_{p*}(\mathcal{P}_*W', \psi')$, where $\psi' - d\mathbf{b}$ is logarithmic for some $\mathbf{b} \in \tau_p^{-1}\mathbb{C}[\tau_p^{-1}]$ with $\deg_{\tau_p^{-1}} \mathbf{b} = m < p$. If $p = 1$, we assume that any eigenvalue of $\text{Res}(\psi')$ is not 0. The claim of Theorem 5.26 follows from the next proposition.

Proposition 5.28 *There exists $(\mathcal{P}_*V', \theta')$ on $U_{\zeta_{p-m}}$ such that (i) $\theta' - d\mathbf{a}$ is logarithmic for some $\mathbf{a} \in \zeta_{p-m}^{-1}\mathbb{C}[\zeta_{p-m}^{-1}]$, (ii) we have an isomorphism $\varphi_{p-m*}(\mathcal{P}_*V', \theta') \simeq \mathcal{N}_{*}^{\infty,0}(\mathcal{P}_*W, \psi)$.*

Proof To simplify the notation, we set $\eta := \tau_p$ and $u := \zeta_{p-m}$. We have the expression

$$\psi' = (G(\eta) \text{id} + \eta^p f) \varphi_p^*(-\tau^{-2} d\tau),$$

such that (i) $G(\eta) = \sum_{j=1}^m \beta_j \eta^{p-j}$ with $\beta_m \neq 0$, (ii) f is an endomorphism of \mathcal{P}_*W' . We fix a holomorphic function $\eta^{(0)}(u)$ such that $G(\eta^{(0)}(u)) - u^{p-m} = 0$ such that $0 < C_1 \leq |\eta^{(0)}/u| \leq C_2$ for some constants C_i .

We set $\mathcal{P}_{c+p-m/2}\mathcal{V} := \mathcal{P}_{c+p-m/2} \varphi_{p-m}^* \mathcal{N}_{*}^{\infty,0}(\mathcal{P}_*W, \psi)$. Let \mathbf{v} be a frame of $\mathcal{P}_c W'$ compatible with the parabolic structure. We set $\tilde{v}_{ij} = u^{-i} \eta^i v_j$ ($0 \leq i \leq p-m-1$, $1 \leq j \leq \text{rank } W'$). They induce a frame of $\mathcal{P}_{c+p-m/2}\mathcal{V}$ compatible with the parabolic structure. By using the frame, for any $c-1 < d \leq c$, we obtain an isomorphism $\text{Gr}_{d+p-m/2}^{\mathcal{P}}(\mathcal{V}) \simeq \text{Gr}_d^{\mathcal{P}}(W') \otimes \mathbb{C}^{p-m}$. The following lemma can be checked directly.

Lemma 5.29 *$u^{-1}\eta$ gives an endomorphism F of $\mathcal{P}_*\mathcal{V}$, preserving the parabolic structure, and the induced endomorphism on $\text{Gr}^{\mathcal{P}}(\mathcal{V})$ is given by $F(\tilde{v}_{ij}) = \tilde{v}_{i+1,j}$ ($i = 0, \dots, p-m-2$) and $F(\tilde{v}_{p-m-1,j}) = -\beta_m^{-1} \tilde{v}_{0,j}$. The eigenvalues are $\omega^i \partial_u \eta^{(0)}(0)$ ($i = 0, \dots, p-m-1$), where $\omega = e^{2\pi\sqrt{-1}/(p-m)}$. \blacksquare*

We obtain the decomposition $(\mathcal{P}_*\mathcal{V}, F) = \bigoplus_{i=0}^{p-m-1} (\mathcal{P}_*\mathcal{V}^{(i)}, F^{(i)})$ such that $F_{|0}^{(i)}$ has a unique eigenvalue $\omega^i \partial_u \eta^{(0)}(0)$. We have an isomorphism $\varphi_{p-m*}(\mathcal{P}_*\mathcal{V}^{(0)}, -\tau^{-1} d\zeta) \simeq \mathcal{N}_{*}^{\infty,0}(\mathcal{P}_*W, \psi)$. We also have an isomorphism $\text{Gr}_{c+p-m/2}^{\mathcal{P}}(\mathcal{V}^{(0)}) \simeq \text{Gr}_c^{\mathcal{P}}(W')$.

We have $G(\eta) - u^{p-m} = -\eta^p f$ on \mathcal{V} . Note that $u^{-(p-m-1)} \sum_{j=1}^m \beta_j Q_{p-j-1}(\eta^{(0)}(u), \eta)$ is invertible on $\mathcal{P}_{c+p-m/2}\mathcal{V}^{(0)}$. Hence, we obtain the following on $\mathcal{P}_{c+p-m/2}\mathcal{V}^{(0)}$:

$$u^{p-m-1}(\eta^{(0)}(u) - \eta) = \eta^p f \cdot \left(\sum_{j=1}^m \beta_j Q_{p-j-1}(\eta^{(0)}(u), \eta) \right)^{-1} u^{p-m-1}$$

We have the following:

$$u^{p-m}(\eta^{-p} - \eta^{(0)}(u)^{-p}) = f\eta^p Q_{p-1}(\eta^{(0)}(u)^{-1}, \eta^{-1}) \eta^{(0)}(u)^{-1} \eta^{-1} \left(\sum_{j=1}^m \beta_j Q_{p-j-1}(\eta^{(0)}(u), \eta) \right)^{-1} u^{p-m}$$

Hence, we obtain that $u^{p-m}(\eta^{-p} - \eta^{(0)}(u)^{-p})$ is an endomorphism of $\mathcal{P}_*\mathcal{V}^{(0)}$. We set $\mathcal{P}_*V' := \mathcal{P}_*\mathcal{V}^{(0)}$ and $\theta' := -\tau^{-1}\varphi_{p-m}^*d\zeta = -\eta^{-p}(p-m)u^{p-m}(du/u)$. We have $\mathbf{a} \in u^{-1}\mathbb{C}[u^{-1}]$ uniquely determined by the condition $u\partial_u \mathbf{a} = -\eta^{(0)}(u)^{-p}(p-m)u^{p-m}$. Then, $\theta' - d\mathbf{a}$ is logarithmic. Thus, the proof of Proposition 5.28 and 5.26 are finished. \blacksquare

5.4.5 Good filtered bundles

Let $U_\tau \subset \mathbb{P}^1$ be a neighbourhood of ∞ with a coordinate τ such that $\tau(\infty) = 0$. Let \mathcal{P}_*E be a filtered bundle on $(T \times U_\tau, T \times \{\infty\})$ satisfying **(A1)**. We take a lift $\widetilde{\mathcal{S}p}_\infty(\mathcal{P}_*E) \subset \mathbb{C}$ of $\mathcal{S}p_\infty(\mathcal{P}_*E)$. By shrinking U_τ , we obtain the corresponding filtered bundle \mathcal{P}_*V with an endomorphism g on (U_τ, ∞) . The filtered bundle \mathcal{P}_*E is called good, if $(\mathcal{P}_*V, -\tau^{-2}gd\tau)$ is a good filtered Higgs bundle.

We obtain the following corollary from Proposition 5.28.

Corollary 5.30

- Let $(\mathcal{P}_*\mathcal{E}, \theta)$ be a good filtered Higgs bundle on (T^\vee, D) satisfying **(A0)**. Then, $\text{Nahm}_*(\mathcal{P}_*\mathcal{E}, \theta)$ is a good filtered bundle.
- Let \mathcal{P}_*E be a good filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$ satisfying **(A1–3)**. Then, $\text{Nahm}_*(\mathcal{P}_*E)$ is a good filtered Higgs bundle. \blacksquare

5.5 Filtered bundles (Appendix)

Let X be a complex manifold with a smooth hypersurface D . (For simplicity, we restrict ourselves to the case that D is smooth, which we need in this paper.) Let \mathcal{E} be a coherent $\mathcal{O}_X(*D)$ -module. Let $D = \coprod_{i \in \Lambda} D_i$ be the decomposition into the connected components. A filtered sheaf over \mathcal{E} is an increasing sequence of coherent \mathcal{O}_X -submodules $\mathcal{P}_\mathbf{a}\mathcal{E} \subset \mathcal{E}$ indexed by \mathbb{R}^Λ satisfying the following.

- $\mathcal{P}_\mathbf{a}\mathcal{E}|_{X \setminus D} = \mathcal{E}|_{X \setminus D}$.
- On a small neighbourhood U of D_i ($i \in \Lambda$), $\mathcal{P}_\mathbf{a}\mathcal{E}|_U$ depends only on a_i , which we denote by $\mathcal{P}_{a_i}(\mathcal{E}|_U)$.
- $\mathcal{P}_\mathbf{a}\mathcal{E} = \bigcap_{\mathbf{a} < \mathbf{b}} \mathcal{P}_\mathbf{b}\mathcal{E}$, where $\mathbf{a} < \mathbf{b}$ means $a_i < b_i$ for any $i \in \Lambda$.
- We have $\mathcal{P}_{\mathbf{a}+\mathbf{n}}\mathcal{E} = \mathcal{P}_\mathbf{a}\mathcal{E}(\sum n_i D_i)$, where $\mathbf{n} = (n_i) \in \mathbb{Z}^\Lambda$.

For a small neighbourhood U of D_i , we set $\mathcal{P}_a(\mathcal{E})|_{D_i} := \mathcal{P}_a(\mathcal{E}|_U)|_{D_i}$, $\mathcal{P}_{<a}(\mathcal{E}|_U) := \sum_{b < a} \mathcal{P}_b(\mathcal{E}|_U)$, and ${}^i\text{Gr}_a^{\mathcal{P}}(\mathcal{E}) := \mathcal{P}_a(\mathcal{E}|_U)/\mathcal{P}_{<a}(\mathcal{E}|_U)$, which are coherent \mathcal{O}_{D_i} -modules.

A filtered sheaf $\mathcal{P}_*\mathcal{E}$ is called a filtered bundle, if (i) $\mathcal{P}_\mathbf{a}\mathcal{E}$ are locally free \mathcal{O}_X -modules, (ii) ${}^i\text{Gr}_a^{\mathcal{P}}(\mathcal{E})$ are locally free \mathcal{O}_{D_i} -modules.

A morphism of filtered sheaves $\mathcal{P}_*\mathcal{E}_1 \rightarrow \mathcal{P}_*\mathcal{E}_2$ means a morphism of \mathcal{O}_X -modules which is compatible with the filtrations. A subobject $\mathcal{P}_*\mathcal{E}_1 \subset \mathcal{P}_*\mathcal{E}$ is a subsheaf $\mathcal{E}_1 \subset \mathcal{E}$ satisfying $\mathcal{P}_a(\mathcal{E}_1) \subset \mathcal{P}_a(\mathcal{E})$ for any a . It is called strict, if $\mathcal{P}_a(\mathcal{E}_1) = \mathcal{E}_1 \cap \mathcal{P}_a(\mathcal{E})$ for any $a \in \mathbb{R}$.

The direct sum of filtered bundles $\mathcal{P}_*\mathcal{E}_i$ ($i = 1, 2$) is defined as $\mathcal{P}_a(\mathcal{E}_1 \oplus \mathcal{E}_2) = \mathcal{P}_a\mathcal{E}_1 \oplus \mathcal{P}_a\mathcal{E}_2$. The tensor product of filtered bundles is defined as $\mathcal{P}_a(\mathcal{E}_1 \otimes \mathcal{E}_2) = \sum_{b+c \leq a} \mathcal{P}_b(\mathcal{E}_1) \otimes \mathcal{P}_c(\mathcal{E}_2)$. The inner homomorphism is defined as $\mathcal{P}_a\mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2) = \{f \in \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2) \mid f(\mathcal{P}_b\mathcal{E}_1) \subset \mathcal{P}_{b+a}\mathcal{E}_2\}$.

Let $\varphi : (X', D') \rightarrow (X, D)$ be a ramified covering with $D' = \coprod_{i \in \Lambda} D'_i$ and $D = \coprod_{i \in \Lambda} D_i$. Let e_i be the degree of the ramification. Let $\mathcal{P}_*\mathcal{E}$ be a filtered bundle over \mathcal{E} . The pull back of a filtered bundle is defined by $\mathcal{P}_a\varphi^*\mathcal{E} = \sum_{\mathbf{e}\mathbf{b} + \mathbf{n} \leq \mathbf{a}} \varphi^*(\mathcal{P}_{\mathbf{b}}\mathcal{E}) \otimes \mathcal{O}_{X'}(\sum n_i D'_i)$, where $\mathbf{e}\mathbf{b} = (e_i b_i \mid i \in \Lambda)$.

5.5.1 The parabolic first Chern class

We set $\mathcal{P}ar(\mathcal{P}_*\mathcal{E}, i) := \{a \in \mathbb{R} \mid {}^i\mathrm{Gr}_a^{\mathcal{P}}(\mathcal{E}) \neq 0\}$ and $\mathcal{P}ar(\mathcal{P}_a\mathcal{E}, i) := \{a \in \mathcal{P}ar(\mathcal{P}_*\mathcal{E}, i) \mid a_i - 1 < a \leq a_i\}$. For $a - 1 < b \leq a$, let $F_b\mathcal{P}_a(\mathcal{E})|_{D_i}$ be the image of $\mathcal{P}_b(\mathcal{E})|_{D_i} \rightarrow \mathcal{P}_a(\mathcal{E})|_{D_i}$. We have $\mathrm{Gr}_b^F(\mathcal{P}_a(\mathcal{E})|_{D_i}) = {}^i\mathrm{Gr}_b^{\mathcal{P}}(\mathcal{E})$ for $a - 1 < b \leq a$.

Let $\mathcal{P}_*\mathcal{E}$ be a filtered sheaf on (X, D) . Suppose that \mathcal{E} is torsion-free. The parabolic first Chern class of $\mathcal{P}_*\mathcal{E}$ is defined as

$$\mathrm{par}\text{-}c_1(\mathcal{P}_*\mathcal{E}) = c_1(\mathcal{P}_a\mathcal{E}) - \sum_{i \in \Lambda} \sum_{b \in \mathcal{P}ar(\mathcal{P}_a\mathcal{E}, i)} b \dim {}^i\mathrm{Gr}_b^{\mathcal{P}}(\mathcal{E}) [D_i].$$

Here, $[D_i]$ is the cohomology class of D_i . It is independent of the choice of \mathbf{a} .

Let U_i be a small neighbourhood of D_i . Suppose that we are given a decomposition $\mathcal{P}_*\mathcal{E}|_{U_i} = \bigoplus_{k \in I(i)} \mathcal{P}_*\mathcal{E}_{i,k}$ for each $i \in \Lambda$. Let \mathcal{U} be a locally free \mathcal{O}_X -submodule of \mathcal{E} such that $\mathcal{U}|_{U_i} = \bigoplus_{k \in I(i)} \mathcal{P}_{a(k,i)}\mathcal{E}_{i,k}$, where $a(k, i) \in \mathbb{R}$. It is easy to check the following equality:

$$\mathrm{par}\text{-}c_1(\mathcal{P}_*\mathcal{E}) = c_1(\mathcal{U}) - \sum_{i \in \Lambda} \sum_{k \in I(i)} \sum_{b \in \mathcal{P}ar(\mathcal{P}_{a(k,i)}\mathcal{E}_{i,k})} b \mathrm{rank} \mathrm{Gr}_b^{\mathcal{P}}(\mathcal{E}_{i,k}) [D_i]$$

6 L^2 -instantons and wild harmonic bundles

6.1 Nahm transform for wild harmonic bundles on T^\vee

6.1.1 Construction

Let D be a non-empty finite subset of T^\vee . We fix a Kähler metric $g_{T^\vee \setminus D}$ of $T^\vee \setminus D$, which is Poincaré like around D . Let $(E, \bar{\partial}_E, \theta, h)$ be a wild harmonic bundle on (T^\vee, D) . We assume that $(E, \bar{\partial}_E, \theta, h)$ has a singularity at each point of D , i.e., θ has a pole, or the parabolic structure is non-trivial. Let $H_{L^2}^i(E, \bar{\partial}_E, \theta, h)$ denote the i -th L^2 -cohomology group of $(E, \bar{\partial}_E, \theta, h)$. As recalled in Lemma 5.1, they are isomorphic to the hypercohomology groups of the complex $\mathcal{C}^\bullet(\mathcal{P}_*E \otimes \Omega^\bullet, \theta)$, where (\mathcal{P}_*E, θ) is the associated good filtered Higgs bundle. In particular, they are finite dimensional, and isomorphic to the space of L^2 -harmonic i -forms of $(E, \bar{\partial}_E, \theta, h)$. We have $H_{L^2}^0(E, \bar{\partial}_E, \theta, h) = H_{L^2}^2(E, \bar{\partial}_E, \theta, h) = 0$ unless $(E, \bar{\partial}_E, \theta, h)$ is \mathcal{O}_{T^\vee} with the trivial metric and the trivial Higgs field.

For any (z, w) , let $\mathcal{L}_{z,w}$ denote the harmonic bundle of rank one given by $(\mathbb{C}, \bar{\partial} + z d\bar{\zeta})$ with trivial metric and the Higgs field $w d\zeta$. Let $(E, \bar{\partial}_{E,z}, \theta_w, h)$ denote $(E, \bar{\partial}_E, \theta, h) \otimes \mathcal{L}_{z,w}$. Let $\mathrm{Nahm}(E, \bar{\partial}_E, \theta, h)_{(z,w)}$ be the space of L^2 -harmonic 1-forms of $(E, \bar{\partial}_{E,z}, \theta_w, h)$. It is independent of the choice of the Poincaré like metric $g_{T^\vee \setminus D}$. It is finite dimensional, and naturally isomorphic to $\mathrm{Nahm}(\mathcal{P}_*E, \theta)_{(z,w)}$. It is naturally equipped with the metric h_1 induced by h .

Let $A^{p,q}(E)$ denote the space of L^2 -sections of $E \otimes \Omega_{T^\vee \setminus D}^{p,q}$. Let $\bar{\partial}_{E,z}^* : A^{p,q} \rightarrow A^{p,q-1}$ denote the adjoint of $\bar{\partial}_{E,z} : A^{p,q} \rightarrow A^{p,q+1}$. Let $\theta_w^\dagger : A^{p,q} \rightarrow A^{p-1,q}$ denote the adjoint of $\theta_w : A^{p,q} \rightarrow A^{p+1,q}$. We have $\bar{\partial}_{E,z}^* := -\sqrt{-1}[\Lambda, \partial_E - \bar{z}d\bar{\zeta}]$ and $\theta_w^* = -\sqrt{-1}[\Lambda, \theta_w^\dagger] = -\sqrt{-1}[\Lambda, \theta^\dagger + \bar{w}d\bar{\zeta}]$.

We set $S^+ := A^{0,0}(E) \oplus A^{1,1}(E)$ and $S^- := A^1(E) = A^{0,1}(E) \oplus A^{1,0}(E)$. Let $\mathcal{D}_{z,w} := \bar{\partial}_{E,z} + \theta_w + \bar{\partial}_{E,z}^* + \theta_w^*$ be the closed operator $S^+ \rightarrow S^-$, and let $\mathcal{D}_{z,w}^* := \bar{\partial}_{E,z} + \theta_w + \bar{\partial}_{E,z}^* + \theta_w^*$ denote its adjoint $S^- \rightarrow S^+$. We have $\mathrm{Ker} \mathcal{D}_{z,w}^* = \mathrm{Nahm}(E, \bar{\partial}_E, \theta, h)_{(z,w)}$. (See [38].) By the vanishing $H_{L^2}^i(E, \bar{\partial}_{E,z}, \theta_w, h)$ ($i = 0, 2$), we obtain that \mathcal{D}^* is surjective. Hence, the family $\bigcup_{(z,w)} \mathrm{Nahm}(E, \bar{\partial}_E, \theta, h)_{(z,w)}$ gives a C^∞ -vector bundle on $T \times \mathbb{C}$. (See

[13].) It is equipped with an induced C^∞ -metric h_1 and an induced unitary connection ∇_1 . Because the C^∞ -bundle is also constructed as a family of the cohomology of the complexes $(A^\bullet(E), \bar{\partial}_{E,z} + \theta_w d\zeta)$, it is equipped with a naturally induced holomorphic structure, which is equal to the $(0,1)$ -part of ∇_1 . By the construction, the holomorphic bundle is naturally isomorphic to $\text{Nahm}(\mathcal{P}_*E, \theta)_{|T \times \mathbb{C}}$. (See §6.2.2 for more details on this isomorphism.) We shall give the proof of the following theorem in §6.1.4 after preliminaries.

Theorem 6.1 $(\text{Nahm}(E, \bar{\partial}_E, \theta, h), h_1, \nabla_1)$ is an L^2 -instanton.

We give a remark on the proof. It is rather easy and standard to prove that $(\text{Nahm}(E, \bar{\partial}_E, \theta, h), h_1, \nabla_1)$ is an instanton by using the twistor property of the instanton and harmonic bundles. But, we do not give it in the following. Instead, we follow the argument to use a description of the curvature $F(\nabla_1)$ in terms of the Green operator, which is also standard. Because we need an estimate for the decay of $F(\nabla_1)$, we need the description, anyway.

6.1.2 Preliminary

Let X be a torus \mathbb{C}_ζ/L . Let $D \subset X$ be a finite set. Let $g_A = A d\zeta d\bar{\zeta}$ be a Kähler metric of $X \setminus D$ for some positive valued function A , which is Poincaré like around D . Let $(E, \bar{\partial}_E, \theta, h)$ be a wild harmonic bundle on $X \setminus D$. We set $\mathcal{D} := \bar{\partial}_E + \theta$. Let \mathcal{D}_A^* (resp \mathcal{D}_1^*) denote the formal adjoint of \mathcal{D} with respect to h and g_A (resp. $d\zeta d\bar{\zeta}$). We set $\Delta_A = \mathcal{D}_A^* \mathcal{D}$ and $\Delta_1 = \mathcal{D}_1^* \mathcal{D}$. We have $\Delta_A = A^{-1} \Delta_1$.

Lemma 6.2 Let φ be a section of E on $X \setminus D$ such that

$$\int |\varphi|_h^2 A |d\zeta d\bar{\zeta}| + \int |\Delta_1 \varphi|_h^2 |d\zeta d\bar{\zeta}| < \infty.$$

Then, we have the following finiteness:

$$\int |\varphi|_h^2 |d\zeta d\bar{\zeta}| + \int |\Delta_A \varphi|_h^2 A |d\zeta d\bar{\zeta}| < \infty \quad (67)$$

$$\int h(\varphi, \Delta_1 \varphi) |d\zeta d\bar{\zeta}| = \int h(\varphi, \Delta_A \varphi) A |d\zeta d\bar{\zeta}| = \int |\mathcal{D} \varphi|_h^2 < \infty \quad (68)$$

Proof The finiteness (67) is clear. In (68), the first equality is trivial. The second equality and finiteness can be shown by an argument in the proof of Lemma 4.14. \blacksquare

We set $\mathcal{D}^\dagger := \bar{\partial}_E + \theta^\dagger$. Let $(\mathcal{D}^\dagger)_A^*$ (resp. $(\mathcal{D}^\dagger)_1^*$) denote the formal adjoint of \mathcal{D}^\dagger with respect to g_A (resp. $d\zeta d\bar{\zeta}$). We have $\Delta_1 = (\mathcal{D}^\dagger)_1^* \mathcal{D}^\dagger$ and $\Delta_A = (\mathcal{D}^\dagger)_A^* \mathcal{D}^\dagger$.

Lemma 6.3 Let φ be as in Lemma 6.2. Then, we have

$$\int h(\varphi, \Delta_1 \varphi) |d\zeta d\bar{\zeta}| = \int h(\varphi, \Delta_A \varphi) A |d\zeta d\bar{\zeta}| = \int |\mathcal{D}^\dagger \varphi|_h^2 < \infty. \quad (69)$$

Proof The first equality is trivial. For the second, we have only to apply Lemma 6.2 to a harmonic bundle $(E, \bar{\partial}_E, \theta^\dagger, h)$ on $X \setminus D$. \blacksquare

6.1.3 Estimate

Let X be a torus \mathbb{C}_ζ/L with a non-empty finite subset D . We use the Euclidean metric $d\zeta d\bar{\zeta}$ of X . Let $\text{dvol}_X = |d\zeta d\bar{\zeta}|$ denote the associated volume form. Let $(E, \bar{\partial}_E, \theta, h)$ be a wild harmonic bundle on (X, D) . Assume that the harmonic bundle has a singularity at each point of D .

Let $\nabla_h = \bar{\partial}_E + \partial_E$ be the Chern connection. Let \mathcal{H}_w be the space of the sections of E on $X \setminus D$ such that

$$\int_X |\varphi|_h^2 \text{dvol}_X + \int_X \left(|\nabla_h \varphi|_h^2 + |(\theta + w d\zeta) \varphi|_h^2 \right) < \infty.$$

Proposition 6.4 *There exist positive constants $R > 0$, $C > 0$ and $\rho > 0$ such that, if $|w| > R$, the following holds for any $\varphi \in \mathcal{H}_w$:*

$$\int_X \left(|\nabla_h \varphi|_h^2 + |(\theta + w d\zeta) \varphi|_h^2 \right) \geq C |w|^\rho \int_X |\varphi|_h^2 \, \text{dvol}_X$$

(See also a refined estimate in Proposition 6.8 below.)

Proof We use an argument in §2.4 of [50] with an adjustment to our situation. We use the standard distance on X . We take small neighbourhoods B_P of $P \in D$. There exists $R_1 > 0$ and $C_1 > 0$ such that, if $|w| \geq R_1$, then we have $|(\theta + w d\zeta) \varphi|_h^2 \geq C_1 |w|^2 |\varphi|_h^2 \, \text{dvol}_X$ on $X \setminus \bigcup_{P \in D} B_P$. We have only to show the estimate on each B_P . We may assume $P = 0$, and B_P is an ϵ -ball $B_\epsilon = \{|\zeta| \leq \epsilon\}$.

We have a ramified covering $\psi : (B'_\epsilon, 0) \rightarrow (B_\epsilon, 0)$ given by $\psi(u) = u^p$ such that $\psi^*(E, \bar{\partial}_E, \theta, h)$ is unramified, i.e., we have the decomposition

$$\psi^*(E, \bar{\partial}_E, \theta) = \bigoplus_{\mathbf{a} \in u^{-1}\mathbb{C}[u^{-1}]} (E_{\mathbf{a}}, \bar{\partial}_{E_{\mathbf{a}}}, \theta_{\mathbf{a}}),$$

where the Higgs field $\theta_{\mathbf{a}} - d\mathbf{a} \, \text{id}_{E_{\mathbf{a}}}$ are tame. Let $\ell := \max\{\deg_{u^{-1}} \mathbf{a} \mid E_{\mathbf{a}} \neq 0\}$.

Lemma 6.5 *There exists $R' > 0$, $C'_i > 0$ ($i = 1, 2$) such that $|\theta|_h \geq C'_1 |w| |d\zeta|$ on $B_\epsilon \setminus \{|\zeta| < C'_2 |w|^{-p/(\ell+p)}\}$.*

Proof We have only to estimate each $\theta_{\mathbf{a}}$ on B'_ϵ . Let us consider the case $\mathbf{a} \neq 0$. We set $n := \deg_{u^{-1}} \mathbf{a}$. For each w , we have the solutions $b_i(w)$ ($i = 0, \dots, n+p-1$) of the following equation:

$$\partial_u \mathbf{a}(u) + pwu^{p-1} = 0$$

We have the equality $u^{-p+1} \partial_u \mathbf{a}(u) + pw = \alpha \prod_{i=0}^{n+p-1} (u^{-1} - b_i(w)^{-1})$ for some $\alpha \in \mathbb{C} \setminus \{0\}$. We have

$$\theta_{\mathbf{a}} = \partial_u \mathbf{a} \, \text{id}_{E_{\mathbf{a}}} \, du + g_{\mathbf{a}} \, du,$$

where $|g_{\mathbf{a}}|_h \leq C_1 |u|^{-1}$. We have $R_2 > 0$ and $C_2 > 0$ such that the following holds if $|w| > R_2$:

$$C_2^{-1} \leq |b_i(w)| |w|^{1/(n+p)} \leq C_2$$

We take $C_3 \gg C_2$. We set $\mathcal{W}_1 := \{|u| \leq C_3 |w|^{-1/(n+p)}\}$.

On $B'_\epsilon \setminus \mathcal{W}_1$, we have $|g_{\mathbf{a}}|_h \leq (C_1/C_3) |w|^{1/(n+p)}$. We also have

$$|u^{-1} - b_i(w)^{-1}| \geq |b_i(w)^{-1}| - |u^{-1}| \geq (C_2^{-1} - C_3^{-1}) |w|^{1/(n+p)}$$

for any i , and hence $|u^{-p+1} \partial_u \mathbf{a} + pw| \geq |\alpha| (C_2^{-1} - C_3^{-1})^{n+p} |w|$. Hence, if C_3 is sufficiently larger than C_2 , there exist $R_4 > 0$ and $C_4 > 0$ such that the following holds if $|w| > R_4$:

$$|(\partial_u \mathbf{a} + pwu^{p-1}) \, \text{id}_{E_{\mathbf{a}}} + g_{\mathbf{a}}|_h \geq C_4 |w| |u|^{p-1}$$

Hence, we obtain the desired inequality for the integral over $B'_\epsilon \setminus \mathcal{W}_1$ in the case $\mathbf{a} \neq 0$.

Let us consider the case $\mathbf{a} = 0$. We have the expression $\theta_0 = g_0 \, du$, and $|g_0|_h \leq C_{10} |u|^{-1}$ for some $C_{10} > 0$. We take $C_{11} > C_{10}$, and we consider $\mathcal{W} := \{|u| \leq C_{11} |w|^{-1/p}\}$. On $B'_\epsilon \setminus \mathcal{W}$, we have $|wu^{p-1}| \geq C_{11}^{p-1} |w|^{1/p}$. We also have $|g_0|_h \leq (C_{10}/C_{11}) |w|^{1/p}$. Hence, if C_{11} is sufficiently larger than C_{10} , we have the following for some $C_{12} > 0$:

$$|pwu^{p-1} \, \text{id}_{E_0} + g_0|_h \geq C_{12} |wu^{p-1}|$$

Hence, we obtain the desired inequality in the case $\mathbf{a} = 0$. Thus, the proof of Lemma 6.5 is finished. ■

Let φ be an L^2 -section of E on B_ϵ with respect to dvol_X , such that

$$\int_{B_\epsilon} \left(|\nabla_h \varphi|_h^2 + |(\theta + w d\zeta) \varphi|_h^2 \right) \, \text{dvol}_X < \infty.$$

We set $\mathcal{W}_1 := \{|\zeta| < 2C'_2|w|^{-p/(\ell+p)}\}$ and $\mathcal{W}_2 := \{|\zeta| < C'_2|w|^{-p/(\ell+p)}\}$. We have the following type of Poincaré inequality, i.e., there exist $C''' > 0$ and $R'' > 0$ such that the following holds if $|w| > R''$ (see [7], and (2.12) of [50]):

$$|w|^{2p/(\ell+p)} \int_{\mathcal{W}_1} |\varphi|_h^2 |d\zeta d\bar{\zeta}| \leq C''' \left(\int_{\mathcal{W}_1} |d|\varphi|_h|^2 + |w|^{2p/(n+p)} \int_{\mathcal{W}_1 \setminus \mathcal{W}_2} |\varphi|_h^2 |d\zeta d\bar{\zeta}| \right)$$

There exists C''' such that the right hand side is dominated by

$$C''' \left(\int_{\mathcal{W}_1} |\nabla_h \varphi|_h^2 + \int_{\mathcal{W}_1 \setminus \mathcal{W}_2} |(\theta + wd\zeta)\varphi|_h^2 \right)$$

Thus, the proof of Proposition 6.4 is finished. \blacksquare

Let $\mathcal{D} := \bar{\partial}_E + \theta$. Let \mathcal{D}_1^* denote the adjoint with respect to the Euclidean metric $d\zeta d\bar{\zeta}$. Let $\Delta_1 := \mathcal{D}_1^* \circ \mathcal{D}$. Let $g_{X \setminus D}$ be a Kähler metric of $X \setminus D$ which is Poincaré like around D . Let $\text{dvol}_{X \setminus D}$ be the volume form associated to $g_{X \setminus D}$.

Corollary 6.6 *There exist $\rho > 0$ and $C > 0$ such that the following holds:*

- Let φ be a section of E such that

$$\int |\varphi|_h^2 \text{dvol}_{X \setminus D} + \int |\Delta_1 \varphi|_h^2 \text{dvol}_X < \infty. \quad (70)$$

Then, we have the following inequality:

$$C|w|^\rho \left(\int |\varphi|_h^2 \text{dvol}_X \right)^{1/2} \leq \left(\int |\Delta_1 \varphi|_h^2 \text{dvol}_X \right)^{1/2} \quad (71)$$

(See Corollary 6.10 for a refinement.)

Proof Let $\mathcal{D}^\dagger = \partial_E + \theta^\dagger$. From (70), Lemma 6.2 and Lemma 6.3, we obtain $\int |\mathcal{D}\varphi|_h^2 < \infty$ and $\int |\mathcal{D}^\dagger \varphi|_h^2 < \infty$. By using the same lemmas and Proposition 6.4, we obtain

$$C|w|^\rho \int |\varphi|_h^2 \text{dvol}_X \leq \int |\mathcal{D}\varphi|_h^2 + \int |\mathcal{D}^\dagger \varphi|_h^2 = 2 \int h(\varphi, \Delta_1 \varphi) \text{dvol}_X.$$

Then, the claim of the corollary follows. \blacksquare

6.1.4 Proof of Theorem 6.1

Let $\omega_{T^\vee \setminus D}$ be the Kähler form associated to the metric $g_{T^\vee \setminus D}$. The multiplication of $\omega_{T^\vee \setminus D}$ induces an isomorphism $A^{0,0}(E) \simeq A^{1,1}(E)$. It gives an identification $S^+ \simeq A^{0,0}(E) \otimes \langle 1, \omega_{T^\vee \setminus D} \rangle$, where $\langle 1, \omega_{T^\vee \setminus D} \rangle$ denotes the 2-dimensional vector space generated by 1 and $\omega_{T^\vee \setminus D}$. By a general theory of harmonic bundles, the Laplacian $\mathcal{D}_{zw}^* \mathcal{D}_{zw}$ on S^+ is identified with $\Delta_{zw} \otimes \text{id}$ on $A^{0,0}(E) \otimes \langle 1, \omega_{T^\vee \setminus D} \rangle$, where $\Delta_{zw} := (\bar{\partial}_{E,z}^* + \theta_w^*) \circ (\bar{\partial}_{E,z} + \theta_w)$ on $A^{0,0}(E)$. (See [47]. In this case, it can be easily checked directly.) The Green operator G_{zw} for $\mathcal{D}_{zw}^* \mathcal{D}_{zw}$ is identified with $\mathcal{G}_{zw} \otimes \text{id}$, where \mathcal{G}_{zw} is the Green operator of Δ_{zw} on $A^{0,0}(E)$.

For a differential form τ on T^\vee , let $\mu(\tau)$ be an endomorphism of $\bigoplus A^{p,q}(E)$ given by $\mu(\tau)(\varphi) = \tau \wedge \varphi$. Let $d_{T \times \mathbb{C}}$ denote the trivial connection of the product vector bundle $S^- \times (T \times \mathbb{C})$ over $T \times \mathbb{C}$. We have the following relation for the operators on the space of the sections $T \times \mathbb{C} \longrightarrow S^- \times (T \times \mathbb{C})$:

$$[d_{T \times \mathbb{C}}, \bar{\partial} + z d\bar{\zeta}] = dz \mu(d\bar{\zeta}), \quad [d_{T \times \mathbb{C}}, \theta + w d\zeta] = dw \mu(d\zeta),$$

$$[d_{T \times \mathbb{C}}, (\bar{\partial} + z d\bar{\zeta})^*] = d\bar{z} (\sqrt{-1} \Lambda \circ \mu(d\zeta)), \quad [d_{T \times \mathbb{C}}, (\theta + w d\zeta)^*] = d\bar{w} (-\sqrt{-1} \Lambda \mu(d\bar{\zeta}))$$

We set $\Omega := dz \mu(d\bar{\zeta}) + dw \mu(d\zeta) + d\bar{z} (\sqrt{-1} \Lambda \mu(d\zeta)) + d\bar{w} (-\sqrt{-1} \Lambda \mu(d\bar{\zeta}))$.

Let $F(\nabla_1)$ be the curvature of the transformed bundle $\text{Nahm}(E, \bar{\partial}_E, \theta, h)$ with the metric and the unitary connection. Let P_{zw} denote the orthogonal projection of S^- onto $\text{Nahm}(E, \bar{\partial}_E, \theta, h)_{(z,w)}$. Let ψ_i be sections of $\text{Nahm}(E, \bar{\partial}_E, \theta, h)$. Let $\langle \cdot, \cdot \rangle$ denote the hermitian pairing on $A^{p,q}(E)$ induced by h and $\omega_{T^\vee \setminus D}$. We have the following standard computation:

$$\begin{aligned} \langle \psi_1, F(\nabla_1) \psi_2 \rangle &= \langle \psi_1, P_{zw} \circ d \circ P_{zw} d \psi_2 \rangle = \langle \psi_1, d \circ P_{zw} \circ d \psi_2 \rangle = \langle \psi_1, d \circ (P_{zw} - 1) \circ d \psi_2 \rangle \\ &= -\langle d \psi_1, (P_{zw} - 1) \circ d \psi_2 \rangle = \langle d \psi_1, \mathcal{D}_{zw} \circ G_{zw} \circ \mathcal{D}_{zw}^* d \psi_2 \rangle = \langle \mathcal{D}_{zw}^* d \psi_1, G_{zw} \mathcal{D}_{zw}^* d \psi_2 \rangle \\ &= \langle [d, \mathcal{D}_{zw}^*] \psi_1, G_{zw} [d, \mathcal{D}_{zw}^*] \psi_2 \rangle = \langle \Omega \psi_1, G_{zw} \Omega \psi_2 \rangle \quad (72) \end{aligned}$$

We have the expression $\psi_i = \psi_{i1} d\zeta + \psi_{i2} d\bar{\zeta}$. We have

$$\Omega \psi_1 = dz \psi_{11} d\bar{\zeta} d\zeta + dw \psi_{12} d\zeta d\bar{\zeta} + d\bar{w} \psi_{11} \Lambda(d\bar{\zeta} d\zeta) + d\bar{z} \psi_{12} \Lambda(d\bar{\zeta} d\zeta).$$

Let A be determined by $g_{T^\vee \setminus D} = A d\zeta d\bar{\zeta}$. We have the following:

$$G_{zw} \Omega \psi_2 = dz \mathcal{G}_{zw}(A^{-1} \psi_{21}) A d\bar{\zeta} d\zeta + dw \mathcal{G}_{zw}(A^{-1} \psi_{22}) A d\zeta d\bar{\zeta} + d\bar{w} \mathcal{G}_{zw}(\psi_{21} \Lambda(d\bar{\zeta} d\zeta)) + d\bar{z} \mathcal{G}_{zw}(\psi_{22} \Lambda(d\bar{\zeta} d\zeta)).$$

We have the following:

$$\langle \psi_{11} d\bar{\zeta} d\zeta, A \mathcal{G}_{zw}(A^{-1} \psi_{21}) d\bar{\zeta} d\zeta \rangle = \langle \psi_{11} \Lambda(d\bar{\zeta} d\zeta), \mathcal{G}_{zw}(\psi_{21} \Lambda(d\bar{\zeta} d\zeta)) \rangle = 4 \int (\psi_{11}, \mathcal{G}_{zw}(A^{-1} \psi_{21})) d\text{vol}_{T^\vee} \quad (73)$$

$$\langle \psi_{12} d\bar{\zeta} d\zeta, A \mathcal{G}_{zw}(A^{-1} \psi_{22}) d\bar{\zeta} d\zeta \rangle = \langle \psi_{12} \Lambda(d\bar{\zeta} d\zeta), \mathcal{G}_{zw}(\psi_{22} \Lambda(d\bar{\zeta} d\zeta)) \rangle = 4 \int (\psi_{12}, \mathcal{G}_{zw}(A^{-1} \psi_{22})) d\text{vol}_{T^\vee} \quad (74)$$

From these equalities, we obtain $(dz d\bar{z} + dw d\bar{w}) \wedge \langle \psi_1, F(\nabla_1) \psi_2 \rangle = 0$, which means that $\text{Nahm}(E, \bar{\partial}_E, \theta, h)$ with the induced metric h_1 and connection ∇_1 is an instanton.

Let us show that it is an L^2 -instanton. Let $(\bar{\partial}_{E,z} + \theta_w)_1^*$ denote the formal adjoint of $\bar{\partial}_{E,z} + \theta_w$ with respect to h and $d\zeta d\bar{\zeta}$. We set $\Delta_{zw,1} := (\bar{\partial}_{E,z} + \theta_w)_1^* (\bar{\partial}_{E,z} + \theta_w)$. Because $\Delta_{zw,1} = A \Delta_{zw}$, we have $\Delta_{zw,1}(\mathcal{G}_{zw}(A^{-1} \psi_{21})) = \psi_{21}$. We have

$$\int |\mathcal{G}_{zw}(A^{-1} \psi_{21})|_h^2 d\text{vol}_{T^\vee \setminus D} + \int |\psi_{21}|_h^2 d\text{vol}_{T^\vee} < \infty$$

By Corollary 6.6, we have the following for some $\rho > 0$ and $C > 0$:

$$C|w|^{2\rho} \int_{T^\vee} |\mathcal{G}_{zw}(A^{-1} \psi_{21})|_h^2 d\text{vol}_{T^\vee} < \int_{T^\vee} |\psi_{21}|^2 d\text{vol}_{T^\vee}$$

Hence, we obtain

$$\begin{aligned} |\langle \psi_{11} d\bar{\zeta} d\zeta, A \mathcal{G}_{zw}(A^{-1} \psi_{21}) d\bar{\zeta} d\zeta \rangle| &= |\langle \psi_{11} \Lambda(d\bar{\zeta} d\zeta), \mathcal{G}_{zw}(\psi_{21} \Lambda(d\bar{\zeta} d\zeta)) \rangle| \\ &< C|w|^{-\rho} \left(\int |\psi_{11} d\zeta|_h^2 \right)^{1/2} \left(\int |\psi_{21} d\zeta|_h^2 \right)^{1/2} \quad (75) \end{aligned}$$

We have a similar estimate for $|\langle \psi_{12} d\bar{\zeta} d\zeta, \mathcal{G}_{zw}(\psi_{22}) d\bar{\zeta} d\zeta \rangle| = |\langle \psi_{12} \Lambda(d\bar{\zeta} d\zeta), \mathcal{G}_{zw}(\psi_{22} \Lambda(d\bar{\zeta} d\zeta)) \rangle|$. From those estimates, we obtain $|F(\nabla_1)| = O(|w|^{-\rho})$ for some $\rho > 0$. Because $\text{Nahm}(E, \bar{\partial}_E, \theta, h)|_{T \times \mathbb{C}} \simeq \text{Nahm}(\mathcal{P}_* E, \theta)$, we can apply Theorem 3.20, and hence we obtain that $F(\nabla_1)$ is L^2 . Thus, the proof of Theorem 6.1 is finished. \blacksquare

Remark 6.7 We can prove that the curvature is L^2 directly by using Corollary 6.10 below. \blacksquare

6.1.5 Refined estimates (Appendix)

We refine the estimates in §6.1.3, i.e., we show that ρ can be replaced with $1 + \rho$.

Proposition 6.8 *There exist positive constants $R > 0$, $C > 0$ and $\rho > 0$ such that, if $|w| > R$, the following holds for any $\varphi \in \mathcal{H}_w$:*

$$\int_X \left(|\nabla_h \varphi|_h^2 + |(\theta + w d\zeta)\varphi|_h^2 \right) \geq C |w|^{1+\rho} \int_X |\varphi|_h^2 \, \text{dvol}_X$$

Proof We again use the argument in §2.4 of [50] with an adjustment to our situation. We use the standard distance on X . We take small neighbourhoods B_P of $P \in D$. There exists $R_1 > 0$ and $C_1 > 0$ such that, if $|w| \geq R_1$, then we have $|(\theta + w d\zeta)\varphi|_h^2 \geq C_1 |w|^2 |\varphi|_h^2 \, \text{dvol}_X$ on $X \setminus \bigcup_{P \in D} B_P$. We have only to show the estimate on each B_P . We may assume $P = 0$, and B_P is an ϵ -ball $B_\epsilon = \{|\zeta| \leq \epsilon\}$.

We have a ramified covering $\psi : (B'_\epsilon, 0) \longrightarrow (B_\epsilon, 0)$ given by $\psi(u) = u^p$ such that $\psi^*(E, \bar{\partial}_E, \theta, h)$ is unramified, i.e., we have the decomposition

$$\psi^*(E, \bar{\partial}_E, \theta) = \bigoplus_{\mathbf{a} \in u^{-1}\mathbb{C}[u^{-1}]} (E_{\mathbf{a}}, \bar{\partial}_{E_{\mathbf{a}}}, \theta_{\mathbf{a}}), \quad (76)$$

where the Higgs field $\theta_{\mathbf{a}} - d\mathbf{a} \, \text{id}_{E_{\mathbf{a}}}$ are tame. Let $h' = \bigoplus_{\mathbf{a}} h|_{E_{\mathbf{a}}}$, and let $\nabla_{h'}$ denote the unitary connection associated to $\psi^*(E, \bar{\partial}_E)$ with h' . By the asymptotic orthogonality of the decomposition (76) with respect to h (see [38]), we have the following inequality:

$$\int_{B'_\epsilon} \left(|\nabla_{h'} \varphi|_{h'}^2 + |(\theta + w d\zeta)\varphi|_h^2 \right) \geq C_2 \int_{B'_\epsilon} \left(|\nabla_{h'} \varphi|_{h'}^2 + |(\theta_{\mathbf{a}} + w d\zeta)\varphi|_{h'}^2 \right)$$

$$\int_{B'_\epsilon(P)} |\varphi|_h^2 \psi^* \, \text{dvol}_X \leq C_3 \int_{B'_\epsilon(P)} |\varphi|_{h'}^2 \psi^* \, \text{dvol}_X$$

Hence, we need only the estimate with respect to the metric h' .

Let us begin with the estimate for sections of $E_{\mathbf{a}}$ with $\mathbf{a} \neq 0$. We set $n := \deg_{u^{-1}} \mathbf{a}$.

Lemma 6.9 *There exist constants $R' > 0$ and $C' > 0$ such the following holds if $|w| \geq R'$:*

- Let φ be an L^2 -section of $E_{\mathbf{a}}$ on B'_ϵ with respect to $\psi^* \, \text{dvol}_X$, such that

$$\int_{B'_\epsilon} \left(|\nabla_{h'} \varphi|_{h'}^2 + |(\theta_{\mathbf{a}} + w d\zeta)\varphi|_{h'}^2 \right) \psi^* \, \text{dvol}_X < \infty.$$

Then, we have

$$|w|^e \int_{B'_\epsilon} |\varphi|_{h'}^2 \psi^* \, \text{dvol}_X < C' \int_{B'_\epsilon} \left(|\nabla_{h'} \varphi|_{h'}^2 + |(\theta_{\mathbf{a}} + w d\zeta)\varphi|_{h'}^2 \right) \psi^* \, \text{dvol}_X.$$

Here, $e = 1 + p/(n + p) > 1$.

Proof For each w , we have the solutions $b_i(w)$ ($i = 0, \dots, n + p - 1$) of the following equation:

$$\partial_u \mathbf{a}(u) + pwu^{p-1} = 0$$

We have the equality $u^{-p+1} \partial_u \mathbf{a}(u) + pw = \alpha \prod_{i=0}^{n+p-1} (u^{-1} - b_i(w)^{-1})$ for some $\alpha \in \mathbb{C} \setminus \{0\}$. We have

$$\theta_{\mathbf{a}} = \partial_u \mathbf{a} \, \text{id}_{E_{\mathbf{a}}} \, du + g_{\mathbf{a}} \, du,$$

where $|g_{\mathbf{a}}|_{h'} \leq C_1 |u|^{-1}$. We have $R_2 > 0$ and $C_2 > 0$ such that the following holds if $|w| > R_2$:

$$C_2^{-1} \leq |b_i(w)| |w|^{1/(n+p)} \leq C_2$$

We take $C_3 \gg C_2$. We set $\mathcal{U}_1 := \{|u| \leq C_3^{-1}|w|^{-1/(n+p)}\}$ and $\mathcal{U}_2 := \{|u| \leq C_3|w|^{-1/(n+p)}\}$.

Let us consider the estimate on $B'_\epsilon \setminus \mathcal{U}_2$. We have $|g_\mathbf{a}|_{h'} \leq (C_1/C_3)|w|^{1/(n+p)}$. We also have

$$|u^{-1} - b_i(w)^{-1}| \geq |b_i(w)^{-1}| - |u^{-1}| \geq (C_2^{-1} - C_3^{-1})|w|^{1/(n+p)}$$

for any i , and hence $|u^{-p+1}\partial_u \mathbf{a} + pw| \geq |\alpha|(C_2^{-1} - C_3^{-1})^{n+p}|w|$. Hence, if C_3 is sufficiently larger than C_2 , we have the following for some $C_4 > 0$:

$$|(\partial_u \mathbf{a} + pwu^{p-1})\varphi + g_\mathbf{a}(\varphi)|_{h'} \geq C_4|w| |\varphi|_{h'} |u|^{p-1}$$

Hence, we obtain the inequality for the integral over $B'_\epsilon \setminus \mathcal{U}_2$.

Let us consider the estimate on \mathcal{U}_1 . There exist $C_5 > 0$ and $R_5 > 0$ such that

$$|(\partial_u \mathbf{a} + pwu^{p-1})\varphi du|_{h'} \geq C_5|u|^{-n-1}|\varphi|_{h'} |du|.$$

We also have $|g_\mathbf{a}\varphi du|_{h'} \leq C_1|u^{-1}||\varphi|_{h'} |du|$. Hence, there exists $C_6 > 0$ such that

$$\left|(\theta_\mathbf{a} + wpu^{p-1} du)\varphi\right|_{h'}^2 \geq C_6|\varphi|_{h'}^2|u|^{-2(n+p)}|u|^{2(p-1)}|du d\bar{u}| \geq C_6C_3|\varphi|_{h'}^2|w|^2|u|^{2(p-1)}|du d\bar{u}|.$$

Therefore, we have the desired inequality for the integral over \mathcal{U}_1 .

Let us consider the estimate on $\mathcal{U}_2 \setminus \mathcal{U}_1$. For each $i = 0, \dots, n+p-1$, we set $\tilde{\mathcal{V}}_i := \{|u - b_i(w)| \leq \epsilon_1|w|^{-1/(n+p)}\}$ for some $\epsilon_1 > 0$. Let $u \in \mathcal{U}_2 \setminus (\mathcal{U}_1 \cup \bigcup_i \tilde{\mathcal{V}}_i)$. We have

$$|u^{-p+1}\partial_u \mathbf{a} + pw| = |pw||u|^{-p-n} \prod_{i=0}^{n+p-1} |u - b_i(w)| \geq pC_3^{-1}|w|^2 \prod_{i=0}^{n+p-1} |u - b_i(w)| \geq pC_3^{-1}\epsilon_1^{p+n}|w|$$

We also have the following:

$$|g_\mathbf{a}\varphi|_{h'} \leq C_1|u|^{p-1}|\varphi|_{h'} \cdot |u|^{-p} \leq C_1|u|^{p-1}|\varphi|_{h'} \cdot C_3|w|^{p/(n+p)} = C_1|u|^{p-1}|\varphi|_{h'} |w| \cdot C_3|w|^{-n/(n+p)}$$

Hence, there exists $C_7 > 0$ and $R_7 > 0$ such that the following holds on $\mathcal{U}_2 \setminus (\mathcal{U}_1 \cup \bigcup_i \tilde{\mathcal{V}}_i)$, if $|w| \geq R_7$:

$$|(\partial_u \mathbf{a} + pwu^{p-1})\varphi du + g_\mathbf{a}\varphi du|_{h'} \geq C_7|w| |\varphi|_{h'} |u|^{p-1}|du|$$

We set $a := (n+2)/2(n+p)$. We put $\mathcal{V}_i := \{|u - b_i(w)| \leq \epsilon_1|w|^{-a}\}$ and $\mathcal{V}'_i := \{|u - b_i(w)| \leq \epsilon_1|w|^{-a}/2\}$. On $\tilde{\mathcal{V}}_i \setminus \mathcal{V}'_i$, we have

$$\begin{aligned} |u^{-p+1}\partial_u \mathbf{a} + pw| &= pC_3^{-1}|w|^2 \prod_{i=0}^{n+p-1} |u - b_i(w)| \geq pC_3^{-1}|w|^2(\epsilon_1|w|^{-1/(n+p)})^{n+p-1} \times (\epsilon_1|w|^{-a}/2) \\ &\geq pC_3^{-1}\epsilon_1^{p+n}|w|^{1+1/(n+p)-a} \end{aligned} \quad (77)$$

We also have $|g_\mathbf{a}| \leq C_1C_3|w|^{1/(n+p)}$. Because $-(p-1)/(n+p) + 1 + 1/(n+p) - a > 1/(n+p)$, there exist $C_8 > 0$ and $R_8 > 0$ such that the following holds on $\tilde{\mathcal{V}}_i \setminus \mathcal{V}'_i$, if $|w| \geq R_8$:

$$\left|(\theta_\mathbf{a} + pwu^{p-1} du)\varphi\right|_{h'} \geq C_8|w|^{1+1/(n+p)-a}|u|^{p-1}|du| |\varphi|_{h'} = C_8|w|^{(n+2p)/2(n+p)}|u|^{p-1}|du| |\varphi|_{h'}$$

We have the following type of Poincaré inequality, i.e., there exist $C_9 > 0$ and $R_9 > 0$ such that the following holds on \mathcal{V}_i , if $|w| > R_9$ (see [7] and (2.12) of [50]):

$$|w|^{(n+2)/(n+p)} \int_{\mathcal{V}_i} |\varphi|_{h'}^2 |du d\bar{u}| \leq C_9 \left(\int_{\mathcal{V}_i} |d|\varphi|_{h'}|^2 + |w|^{(n+2)/(n+p)} \int_{\mathcal{V}_i \setminus \mathcal{V}'_i} |\varphi|_{h'}^2 |du d\bar{u}| \right)$$

We also have the following inequalities:

$$|w|^{(n+2p)/(n+p)} \int_{\mathcal{V}_i} |\varphi|_{h'}^2 |u|^{2(p-1)} |du d\bar{u}| \leq C_3^{2(p-1)} |w|^{(n+2)/(n+p)} \int_{\mathcal{V}_i} |\varphi|_{h'}^2 |du d\bar{u}| \quad (78)$$

$$\begin{aligned} |w|^{(n+2)/(n+p)} \int_{\mathcal{V}_i \setminus \mathcal{V}'_i} |\varphi|_{h'}^2 |du d\bar{u}| &\leq C_3^{2(p-1)} |w|^{(n+2p)/(n+p)} \int_{\mathcal{V}_i \setminus \mathcal{V}'_i} |\varphi|_{h'}^2 |u|^{2(p-1)} |du d\bar{u}| \\ &\leq C_8 C_3^{2(p-1)} \int_{\mathcal{V}_i \setminus \mathcal{V}'_i} |(\theta_a + pwu^{p-1} du) \varphi|_{h'}^2 \end{aligned} \quad (79)$$

Then, we obtain the desired inequality for the integral over \mathcal{V}_i . Thus, the proof of Lemma 6.9 is finished. \blacksquare

Let us consider the case $\mathbf{a} = 0$. Because this part is essentially contained in [50], we give just an indication. We take a positive number C_{10} which is sufficiently larger than $|\alpha|$ for any eigenvalues α of the residue of θ_0 . We may assume $|g_0| \leq (C_{10}/10)|\zeta|^{-1}$ on B_ϵ . Take $R_{10} > 0$ sufficiently larger than C_{10} . For $|w| \geq R_{10}$, let $\mathcal{U} := \{|\zeta| \leq C_{10}|w|^{-1}\}$ and $\mathcal{U}' := \{|\zeta| \leq C_{10}|w|^{-1}/2\}$. On $B_\epsilon \setminus \mathcal{U}'$, we have

$$|(\theta_0 + w d\zeta) \varphi|_{h'} \geq |w| |\varphi|_{h'} |d\zeta| - |g_0|_{h'} |\varphi|_{h'} |d\zeta| \geq \frac{4}{5} |w| |\varphi|_{h'} |d\zeta| \quad (80)$$

There exist $C_{11} > 0$ and $R_{11} > 0$ such that the following holds on \mathcal{U} , if $|w| \geq R_{11}$:

$$|w|^2 \int_{\mathcal{U}} |\varphi|_{h'}^2 |d\zeta d\bar{\zeta}| \leq C_{11} \int_{\mathcal{U}} |d|\varphi|_{h'}|^2 + |w|^2 \int_{\mathcal{U} \setminus \mathcal{U}'} |\varphi|_{h'}^2 |d\zeta d\bar{\zeta}| \leq \int_{\mathcal{U}} (C_{11} |\nabla_{h'} \varphi|_{h'}^2 + 4 |(\theta_0 + w d\zeta) \varphi|_{h'}^2) \quad (81)$$

We obtain the desired inequality for sections of E_0 from (80) and (81). Thus, the proof of Proposition 6.8 is finished. \blacksquare

The following is a refinement of Corollary 6.6.

Corollary 6.10 *There exist $\rho > 0$ and $C > 0$ such that the following holds:*

- *Let φ be a section of E such that*

$$\int |\varphi|_h^2 \text{dvol}_{X \setminus D} + \int |\Delta_1 \varphi|_h^2 \text{dvol}_X < \infty. \quad (82)$$

Then, we have the following inequality:

$$C |w|^{1+\rho} \left(\int |\varphi|_h^2 \text{dvol}_X \right)^{1/2} \leq \left(\int |\Delta_1 \varphi|_h^2 \text{dvol}_X \right)^{1/2} \quad (83)$$

Proof It is shown by the argument for Corollary 6.6, by using Proposition 6.8, instead of Proposition 6.4. \blacksquare

6.2 Comparison with the algebraic Nahm transform

6.2.1 Statements

Let $(E, \bar{\partial}_E, \theta, h)$ be a wild harmonic bundle on (T^\vee, D) . Let $\mathcal{P}_* E$ be the associated filtered bundle on (T^\vee, D) . Let (E_1, h_1, ∇_1) be the L^2 -instanton on $T \times \mathbb{C}$ obtained as the Nahm transform of $(E, \bar{\partial}_E, \theta, h)$ (see §6.1). Let $\mathcal{P}_* E_1$ be the associated filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$.

Theorem 6.11 *There is a natural isomorphism of the filtered bundles $\mathcal{P}_* E_1 \simeq \text{Nahm}_*(\mathcal{P}_* E, \theta)$.*

Conversely, let (E_1, ∇_1, h_1) be an L^2 -instanton on $T \times \mathbb{C}$. Let $\mathcal{P}_* E_1$ be the associated filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$. Let $(E, \bar{\partial}_E, \theta, h)$ be the wild harmonic bundle on (T^\vee, D) obtained as the Nahm transform of (E_1, ∇_1, h_1) (see §4.4). Let $(\mathcal{P}_* E, \theta)$ be the associated filtered Higgs bundle.

Theorem 6.12 *There is a natural isomorphism of the filtered Higgs bundles $(\mathcal{P}_*E, \theta) \simeq \text{Nahm}_*(\mathcal{P}_*E_1)$.*

We obtain the following involutivity of the Nahm transform in the following sense.

Corollary 6.13 *For an L^2 -instanton (E_1, ∇_1, h_1) on $T \times \mathbb{C}$, we have an isomorphism*

$$\text{Nahm}(\text{Nahm}(E_1, \nabla_1, h_1)) \simeq (E_1, \nabla_1, h_1).$$

For a wild harmonic bundle $(E, \bar{\partial}_E, \theta, h)$ on T^\vee , we have an isomorphism

$$\text{Nahm}(\text{Nahm}(E, \bar{\partial}_E, \theta, h)) \simeq (E, \bar{\partial}_E, \theta, h).$$

Proof It follows from Proposition 5.21, Theorem 6.11, Theorem 6.12, and the uniqueness of the harmonic metric or Hermitian-Einstein metric adapted to the parabolic structure. (See Proposition 4.5 for the uniqueness of Hermitian-Einstein metric. See [7] for the uniqueness of the harmonic metric. See also [38].) ■

6.2.2 Proof of Theorem 6.11

Let us construct an isomorphism $(E_1, \bar{\partial}_{E_1}) \simeq \text{Nahm}(\mathcal{P}_*E, \theta)|_{T \times \mathbb{C}}$. We recall the monad construction of $E_1 = \text{Nahm}(E, \bar{\partial}_E, \theta, h)$ [13]. We use the notation in §6.1.1. Let $g_{T^\vee \setminus D}$ be a Poincaré like Kähler metric of $T^\vee \setminus D$. Let $\mathcal{A}^i(E, \bar{\partial}_E, \theta, h)$ denote the space of sections φ of $E \otimes \Omega^i$ on $T^\vee \setminus D$ such that φ and $(\bar{\partial}_E + \theta)\varphi$ are L^2 with respect to h and $g_{T^\vee \setminus D}$. Note that the conditions also imply $(\bar{\partial}_{E,z} + \theta_w)\varphi$ are L^2 for any $(z, w) \in T \times \mathbb{C}$. Let $\underline{\mathcal{A}}^i$ denote the sheaf of holomorphic sections of the product bundle $\mathcal{A}^i(E, \bar{\partial}_E, \theta, h) \times (T \times \mathbb{C})$ over $T \times \mathbb{C}$. We have the morphisms $\delta^i : \underline{\mathcal{A}}^i \rightarrow \underline{\mathcal{A}}^{i+1}$ induced by $\bar{\partial}_{E,z} + \theta_w$, and the sheaf of holomorphic sections of E_1 is isomorphic to $\text{Ker } \delta^1 / \text{Im } \delta^0$.

Applying the construction in the proof of Lemma 5.1 around each point of D , we extend E and $E \otimes \Omega^1$ to $\mathcal{C}_{L^2}^0(\mathcal{P}_*E, \theta)$ and $\mathcal{C}_{L^2}^1(\mathcal{P}_*E, \theta)$. Let $\mathcal{C}_{L^2}^{i,\bullet}(\mathcal{P}_*E, \theta)$ denote the Dolbeault resolution of $\mathcal{C}_{L^2}^i(\mathcal{P}_*E, \theta)$.

For $I \subset \{1, 2, 3\}$, let p_I denote the projection of $T^\vee \times T \times \mathbb{C}$ onto the product of the i -th components ($i \in I$). On $T^\vee \times T \times \mathbb{C}$, we set

$$\tilde{\mathcal{C}}_{L^2}^i := \bigoplus_{k+\ell=i} p_1^{-1} \mathcal{C}_{L^2}^{k,\ell}(\mathcal{P}_*E, \theta) \otimes_{p_1^{-1} \mathcal{O}_{T^\vee}} p_{12}^* \mathcal{P}oin$$

We have $\delta^i : \tilde{\mathcal{C}}_{L^2}^i \rightarrow \tilde{\mathcal{C}}_{L^2}^{i+1}$ induced by $\bar{\partial}_{E,z} + \theta_w$. We have a natural inclusion $\Phi : p_{23*} \tilde{\mathcal{C}}_{L^2}^\bullet \rightarrow \underline{\mathcal{A}}^\bullet$ of complexes on $T \times \mathbb{C}$. According to the results in §5.1 of [38], Φ is a quasi-isomorphism. We also have the following natural isomorphisms in $D^b(\mathcal{O}_{T \times \mathbb{C}})$:

$$p_{23*} \tilde{\mathcal{C}}_{L^2} \simeq Rp_{23*} \left(p_1^* (\mathcal{C}_{L^2}^\bullet(\mathcal{P}_*E, \theta)) \otimes p_{12}^* \mathcal{P}oin \right) \overset{\Psi}{\simeq} Rp_{23*} \left(p_1^* (\mathcal{C}^\bullet(\mathcal{P}_*E, \theta)) \otimes p_{12}^* \mathcal{P}oin \right) \simeq \text{Nahm}(\mathcal{P}_*E, \theta)|_{T \times \mathbb{C}}$$

(See Lemma 5.1 for Ψ .) Thus, we obtain the desired isomorphism $E_1 \simeq \text{Nahm}(\mathcal{P}_*E, \theta)|_{T \times \mathbb{C}}$, by which we shall identify them.

Lemma 6.14 *To prove Theorem 6.11, we have only to show $\text{Nahm}_a(\mathcal{P}_*E, \theta) \subset \mathcal{P}_a E_1$ for any a .*

Proof By Proposition 5.15, we have $\deg(\text{Nahm}_*(\mathcal{P}_*E, \theta)) = \deg(\mathcal{P}_*E, \theta) = 0$. By Corollary 4.4, we also have $\deg(\mathcal{P}_*E_1) = 0$. Hence, $\text{Nahm}_a(\mathcal{P}_*E, \theta) \subset \mathcal{P}_a E_1$ implies $\text{Nahm}_a(\mathcal{P}_*E, \theta) = \mathcal{P}_a E_1$. ■

To show $\text{Nahm}_a(\mathcal{P}_*E, \theta) \subset \mathcal{P}_a E_1$, we need an estimate of the upper bound of the norms of sections of $\text{Nahm}_a(\mathcal{P}_*E, \theta)$. We use an argument of scaling in [50]. Because we need only the upper bound, we will not consider more precise estimates for harmonic representatives or their approximation.

Let $U_\tau \subset \mathbb{P}^1$ be a neighbourhood of ∞ with the coordinate $\tau = w^{-1}$. If U_τ is sufficiently small, we have the decomposition $\text{Nahm}_*(\mathcal{P}_*E, \theta) = \bigoplus_{P \in D} \text{Nahm}_*(\mathcal{P}_*E, \theta)_P$ by the spectrum on $T \times U_\tau$. We have the refined decomposition

$$\text{Nahm}_*(\mathcal{P}_*E, \theta)_P = \bigoplus_{\mathbf{o} \in \text{Irr}(\theta, P)} \bigoplus_{\alpha \in \mathbb{C}} \text{Nahm}_*(\mathcal{P}_*E, \theta)_{P, \mathbf{o}, \alpha},$$

according to the decomposition of the filtered Higgs bundle $(\mathcal{P}_*E, \theta) = \bigoplus_{\mathfrak{o} \in \text{Irr}(\theta, P)} \bigoplus_{\alpha \in \mathbb{C}} (\mathcal{P}_*E_{P, \mathfrak{o}, \alpha}, \theta_{P, \mathfrak{o}, \alpha})$ around each $P \in D$. We have only to show that $\text{Nahm}_a(\mathcal{P}_*E, \theta)_{P, \mathfrak{o}, \alpha} \subset \mathcal{P}_a E_1$. We shall argue the case $P = \{0\}$ in the following. The other case can be established similarly. We omit the subscript P . We take a small neighbourhood $U_\zeta \subset T^\vee$ of $\{0\}$.

Let us consider the case $(\mathfrak{o}, \alpha) \neq (0, 0)$. Take $\mathfrak{a} \in \mathfrak{o}$. For each $c \in \mathbb{R}$, we have the frame of $\text{Nahm}_c(\mathcal{P}_*E, \theta)_{P, \mathfrak{o}, \alpha}$ in Lemma 5.24. We have only to show

$$\left| \left[\zeta_{\mathfrak{o}}^j v_{\mathfrak{o}, i} d\zeta_{\mathfrak{o}} / \zeta_{\mathfrak{o}} \right] \right|_{h_1} = O(|w|^{(b-j-m_{\mathfrak{o}}/2)(p_{\mathfrak{o}}+m_{\mathfrak{o}})^{-1}}) \quad (84)$$

Here, b is the parabolic degree of $v_{\mathfrak{o}, i}$.

We give a preliminary. We have the expression $\zeta_{\mathfrak{o}} \partial_{\zeta_{\mathfrak{o}}} \mathfrak{a} + p_{\mathfrak{o}} \alpha = \sum_{j=0}^{m_{\mathfrak{o}}} \alpha_j \zeta_{\mathfrak{o}}^{-j} =: G(\zeta_{\mathfrak{o}})$. We fix a complex number γ such that $\alpha_{m_{\mathfrak{o}}} + p_{\mathfrak{o}} \gamma^{p_{\mathfrak{o}}+m_{\mathfrak{o}}} = 0$. Take a covering $U_\eta \rightarrow U_\tau$ given by $\eta \mapsto \eta^{p_{\mathfrak{o}}+m_{\mathfrak{o}}}$. If U_τ is sufficiently small, we can take holomorphic functions $u_0^{(i)}(\xi)$ ($i = 1, \dots, p_{\mathfrak{o}}+m_{\mathfrak{o}}$) satisfying the following:

$$G(u_0^{(i)}(\eta)) + p_{\mathfrak{o}} u_0^{(i)}(\eta)^{p_{\mathfrak{o}}} \eta^{-p_{\mathfrak{o}}-m_{\mathfrak{o}}} = 0, \quad \lim_{\xi \rightarrow \infty} u_0^{(i)}(\eta)/\eta = \gamma \exp(2\pi\sqrt{-1}i/(m_{\mathfrak{o}} + p_{\mathfrak{o}}))$$

There exist $C_1 > 0$ and $\epsilon_1 > 0$ such that

$$\left| \eta^{-1} u_0^{(i)}(\eta) - \gamma \exp(2\pi\sqrt{-1}i/(m_{\mathfrak{o}} + p_{\mathfrak{o}})) \right| \leq C_1 |\eta|^{\epsilon_1}.$$

Lemma 6.15 *Let Z_η denote the support of $\text{Cok}(\eta^{p_{\mathfrak{o}}+m_{\mathfrak{o}}} \zeta_{\mathfrak{o}}^{m_{\mathfrak{o}}} \theta_{\mathfrak{a}, \alpha}^{\mathfrak{o}} + p_{\mathfrak{o}} \zeta_{\mathfrak{o}}^{p_{\mathfrak{o}}+m_{\mathfrak{o}}} d\zeta_{\mathfrak{o}} / \zeta_{\mathfrak{o}})$ on $U_{\zeta_{\mathfrak{o}}, \eta}$. If U_ζ and U_τ are sufficiently small, there exists a decomposition $Z_\eta = \coprod_{i=1}^{p_{\mathfrak{o}}+m_{\mathfrak{o}}} Z_\eta^{(i)}$ such that the following holds for any $u \in Z_\eta^{(i)}$:*

$$|u_0^{(i)}(\eta) - u| \leq C |\eta|^{1+m_{\mathfrak{o}}+\epsilon}$$

Here C and ϵ are positive constants which are independent of η .

Proof Take $u_1 \in Z_\eta$. There exists a possibly multi-valued holomorphic 1-form $\nu(\zeta_{\mathfrak{o}}) d\zeta_{\mathfrak{o}} / \zeta_{\mathfrak{o}}$ obtained as the eigenvalue of $\theta_{\mathfrak{a}}^{\mathfrak{o}}$, such that $\nu(u_1) + \eta^{-p_{\mathfrak{o}}-m_{\mathfrak{o}}} p_{\mathfrak{o}} u_1^{p_{\mathfrak{o}}} = 0$. Because $\nu(\zeta_{\mathfrak{o}}) - G(\zeta_{\mathfrak{o}}) = O(\zeta_{\mathfrak{o}}^\epsilon)$, there exist $C_2 > 0$ and $\epsilon_2 > 0$, independently from η , such that the following holds for some unique i :

$$\left| \eta^{-1} u_1 - \gamma \exp(2\pi\sqrt{-1}i/(p_{\mathfrak{o}} + m_{\mathfrak{o}})) \right| \leq C_2 |\eta|^{\epsilon_2}. \quad (85)$$

We obtain the decomposition of Z_η by (85).

Let $u_1 \in Z_\eta^{(i)}$. We set $Q_q(x, y) := \sum_{i+j=q} x^i y^j$. We have

$$\begin{aligned} & \left((u_0^{(i)}(\eta)/\eta)^{-1} - (u_1/\eta)^{-1} \right) \times \\ & \quad \left(\sum_{j=1}^{m_{\mathfrak{o}}} \alpha_j \eta^{m_{\mathfrak{o}}-j} Q_{j-1}((u_0^{(i)}/\eta)^{-1}, (u_1/\eta)^{-1}) - (u_0^{(i)}/\eta)(u_1/\eta) p_{\mathfrak{o}} Q_{p_{\mathfrak{o}}-1}(u_0^{(i)}/\eta, u_1/\eta) \right) \\ & \quad = O(|u_1/\eta|^\epsilon |\eta|^{m_{\mathfrak{o}}+\epsilon}) \end{aligned} \quad (86)$$

We obtain $|(u_0^{(i)}(\eta)/\eta)^{-1} - (u_1/\eta)^{-1}| = O(|\eta|^{m_{\mathfrak{o}}+\epsilon})$. Then, we obtain the desired estimate. \blacksquare

Let ρ be an $\mathbb{R}_{\geq 0}$ -valued function on \mathbb{C}_η such that $\rho(\eta) = 1$ for $|\eta| < 1/2$ and $\rho(\eta) = 0$ for $|\eta| > 1$. We set $u_0 := u_0^{(0)}$. We consider the following C^∞ -sections of $E_{\mathfrak{a}, \alpha}^{\mathfrak{o}} \otimes \Omega_{X^{\mathfrak{o}}}^1$:

$$\mu_1(v_{\mathfrak{o}, i}, \xi) := \rho(|\xi|^{1+m_{\mathfrak{o}}/2} (\zeta_{\mathfrak{o}} - u_0(\xi))) v_{\mathfrak{o}, i} d\zeta_{\mathfrak{o}} / \zeta_{\mathfrak{o}}$$

$$\mu_2(v_{\mathfrak{o}, i}, \xi) := (\theta_{\mathfrak{a}}^{\mathfrak{o}} + \xi^{p_{\mathfrak{o}}+m_{\mathfrak{o}}} d\zeta_{\mathfrak{o}}^{p_{\mathfrak{o}}})^{-1} \left(\bar{\partial}_E \mu_1(v_{\mathfrak{o}, i}, \xi) \right)$$

By Lemma 6.15, if $|\xi|$ is sufficiently large, $\rho(|\xi|^{1+m_\bullet/2}(\zeta_\bullet - u_0(\xi)))$ is constantly 1 around Z_ξ . Hence, the tuple $\mu(v_{\bullet,i}, \xi) = (\mu_1(v_{\bullet,i}, \xi), \mu_2(v_{\bullet,i}, \xi))$ gives a representative of $[v_{\bullet,i} d\zeta_\bullet / \zeta_\bullet]$.

By an elementary change of variables, we obtain the following for any $\delta > 0$:

$$\int |\mu_1(v_{\bullet,i}, \xi)|_h^2 \leq \int \rho(|\xi|^{1+m_\bullet/2}(\zeta_\bullet - u_0(\xi)))^2 |\zeta_\bullet|^{-2(b+\delta)-2} |d\zeta_\bullet d\bar{\zeta}_\bullet| \leq C_{1\delta} |\xi|^{2(b+\delta)-m_\bullet}$$

Note that we have $|\zeta_\bullet - u_0(\xi)| \sim |\xi|^{-1-m_\bullet/2}$ for ζ_\bullet such that $\bar{\partial}\rho(|\xi|^{1+m_\bullet/2}(\zeta_\bullet - u_0(\xi))) \neq 0$. Hence, we also have the following:

$$\begin{aligned} \int |\mu_2(v_{\bullet,i}, \xi)|_h^2 &\leq C_{2\delta} \int |\bar{\partial}\rho(|\xi|^{1+m_\bullet/2}(\zeta_\bullet - u_0(\xi)))|^2 |\zeta_\bullet|^{-2(b+\delta)} \frac{1}{\left| \zeta_\bullet \partial_{\zeta_\bullet} \mathbf{a} + p_\bullet \alpha + p_\bullet \xi^{p_\bullet+m_\bullet} \zeta_\bullet^{p_\bullet} \right|^2} \\ &\leq C_{3\delta} |\xi|^{2(b+\delta)-2(m_\bullet+1)+2(1+m_\bullet/2)} = C_{3\delta} |\xi|^{2(b+\delta)-m_\bullet} \end{aligned} \quad (87)$$

By the construction of h_1 , we have $|[v_{\bullet,i} d\zeta_\bullet / \zeta_\bullet]|_{h_1}^2 \leq \int (|\mu_1(v_{\bullet,i}, \xi)|_h^2 + |\mu_2(v_{\bullet,i}, \xi)|_h^2)$. Hence, we obtain the desired estimate (84) for $[v_{\bullet,i} d\zeta_\bullet / \zeta_\bullet]$. We obtain the estimate for $[v_{\bullet,i} \zeta^j d\zeta_\bullet / \zeta_\bullet]$ similarly.

Let us consider the case $(\bullet, \alpha) = (\{0\}, 0)$. The following lemma is easy to see.

Lemma 6.16 *Let Z_w denote the support of $\text{Cok}(\theta_{0,0} + w d\zeta)$. There exist $C > 0$ and $\epsilon > 0$ such that $|u| \leq C|w|^{-1-\epsilon}$ holds for any $u \in Z_w$.* \blacksquare

For a holomorphic section s of $\mathcal{C}^1(\mathcal{P}_* E_{P, \{0\}, 0} \otimes \Omega^\bullet, \theta)$ (see §5.1.2), we consider the following C^∞ -sections of $E_{P, \{0\}, 0} \otimes \Omega^1$:

$$\begin{aligned} \mu_1(s, w) &:= (\rho(\zeta) - \rho(w\zeta)) s d\zeta / \zeta, & \mu_2(s, w) &:= (\theta_{P, \{0\}, 0} + w d\zeta)^{-1} (\bar{\partial}\mu_1(s)) \\ \mu'_1(s, w) &:= \rho(\zeta) s d\zeta / \zeta, & \mu'_2(s, w) &:= (\theta_{P, \{0\}, 0} + w d\zeta)^{-1} (\bar{\partial}\mu'_1(s)) \end{aligned}$$

By Lemma 6.16, μ_2 and μ'_2 are well defined. The tuples $\mu(s, w) = (\mu_1(s, w), \mu_2(s, w))$ and $\mu'(s, w) = (\mu'_1(s, w), \mu'_2(s, w))$ naturally induce the same holomorphic section of $\text{Nahm}(\mathcal{P}_* E)_P$. If s is a section of $\mathcal{P}_c E_{\{0\}, 0}$, then it is elementary to show the following for any $\delta > 0$:

$$\int |\mu_i(s, w)|_h^2 \leq C_\delta |w|^{2(c+\delta)}$$

We obtain $|\mu'(s, w)|_{h_1} \leq C|w|^{c+\delta}$ for any $\delta > 0$. By the construction of $\text{Nahm}_*(\mathcal{P}_* E)_{P, \{0\}, 0}$, we obtain $\text{Nahm}_*(\mathcal{P}_* E)_{P, \{0\}, 0} \subset \mathcal{P}_* E_1$. Thus, the proof of Theorem 6.11 is finished. \blacksquare

6.2.3 Proof of Theorem 6.12

Let us construct an isomorphism of the Higgs bundles $(E, \bar{\partial}_E, \theta) \simeq \text{Nahm}(\mathcal{P}_* E_1)_{|T^\vee \setminus D}$. Let us recall the monad construction of $\text{Nahm}(E_1, \nabla_1)$. Let $\mathcal{A}^{0,i}$ denote the space of sections φ of $E_1 \otimes \Omega^{0,i}$ on $T \times \mathbb{C}$, such that φ and $\bar{\partial}_{E_1} \varphi$ are L^2 with respect to h_1 and the Euclidean metric. Let $\underline{\mathcal{A}}^{0,i}$ denote the sheaf of holomorphic sections of the product bundle $\mathcal{A}^{0,i} \times (T^\vee \setminus D)$ over $T^\vee \setminus D$. We have the morphism $\delta^i : \underline{\mathcal{A}}^{0,i} \rightarrow \underline{\mathcal{A}}^{0,i+1}$ induced by $\bar{\partial}_{E_1} - \zeta d\bar{z}$, and the sheaf of holomorphic sections of $(E, \bar{\partial}_E)$ is isomorphic to $\text{Ker } \delta^1 / \text{Im } \delta^0$.

Let $\mathcal{A}_c^{0,i}$ denote the space of C^∞ -sections of $E_1 \otimes \Omega^{0,i}$. Similarly, we obtain a complex $(\underline{\mathcal{A}}_c^{0,\bullet}, \delta)$ on $T^\vee \setminus D$. The natural inclusion $\mathcal{A}_c^{0,\bullet} \rightarrow \mathcal{A}^{0,\bullet}$ is a quasi-isomorphism by Proposition 4.8. For $I \subset \{1, 2, 3\}$, let p_I denote the projection of $T^\vee \times T \times \mathbb{C}$ onto the product of the i -th components. We have a natural quasi-isomorphism $\underline{\mathcal{A}}_c^{0,\bullet} \rightarrow R p_{1*}(p_{23}^* \mathcal{P}_a E_1 \otimes p_{12}^* \mathcal{P}oin^{-1})$. Hence, we obtain a holomorphic isomorphism $E \simeq \text{Nahm}(\mathcal{P}_* E_1)_{|T^\vee \setminus D}$, by which we identify them. The Higgs fields are equal, because they are induced by the multiplication of $-w$.

We give a preliminary. Let $U \subset \mathbb{P}^1$ be a small neighbourhood of ∞ . On $T \times U$, we have the following decomposition

$$\mathcal{P}_* E_1 = \bigoplus_{P \in Sp_\infty(E_1)} \bigoplus_{\bullet, \alpha} \mathcal{P}_*(E_1)_{P, \bullet, \alpha}. \quad (88)$$

Fix a lift of $\mathcal{S}p_\infty(E_1) \subset T^\vee$ to $\widetilde{\mathcal{S}p}_\infty(E_1) \subset \mathbb{C}$. We have the filtered bundles with an endomorphism (\mathcal{P}_*V, g) on U , corresponding to \mathcal{P}_*E_1 . It has a decomposition $(\mathcal{P}_*V, g) = \bigoplus (\mathcal{P}_*V_{P, \mathfrak{o}, \alpha}, g_{P, \mathfrak{o}, \alpha})$.

Let $\mathcal{U} \subset \mathcal{P}_{-1}(E_1)$ be the subsheaf such that $\mathcal{U}|_{T \times \mathbb{C}} = \mathcal{P}_{-1}(E_1)|_{T \times \mathbb{C}}$ and

$$\mathcal{U} = \bigoplus_P \left(\mathcal{P}_{-1}(E_1)_{P, \{0\}, 0} \oplus \bigoplus_{(\mathfrak{o}, \alpha) \neq (\{0\}, 0)} \mathcal{P}_{<-1}(E_1)_{P, \mathfrak{o}, \alpha} \right) \quad (89)$$

around $T \times \{\infty\}$. We use the notation in §5.3.2.

Lemma 6.17 *We have $N(\mathcal{U}) \subset \mathcal{P}_0E$.*

Proof We give an argument around $0 \in T^\vee$, by supposing $0 \in D$. The other case can be proved similarly. We may suppose the lift of $0 \in D$ is $0 \in \mathbb{C}$. Let t be a holomorphic section of $N(\mathcal{U})$ around $0 \in T^\vee$. We have to show $|t|_h = O(|\zeta|^{-\delta})$ for any $\delta > 0$. It is represented by a family of C^∞ -sections $\kappa(\zeta) = \kappa^1(\zeta)d\bar{z} + \kappa^2(\zeta)d\bar{w}$ of $\mathcal{P}_{-1}E_1 \otimes \Omega_{T \times \mathbb{P}^1}^{0,1} \otimes L_\zeta^{-1}$. According to the decomposition (89), we have

$$\kappa^i(\zeta) = \sum_{P, \mathfrak{o}, \alpha} \kappa^i(\zeta)_{P, \mathfrak{o}, \alpha}.$$

If $P \neq 0$, we may assume $\kappa^i(\zeta)_{P, \mathfrak{o}, \alpha} = 0$ on U . (See the proof of Proposition 4.7.) Let $\text{dvol} := |dzd\bar{z}dw d\bar{w}|$.

We take a C^∞ -metric h_2 of \mathcal{U} . Note $h_1 \leq h_2|w|^{-2+\delta}$ for any $\delta > 0$ on $\mathcal{P}_{-1}(E_1)_{P, \{0\}, 0}$, and $h_1 \leq h_2|w|^{-1-\epsilon}$ for some $\epsilon > 0$ on $\mathcal{P}_{-1}(E_1)_{P, \mathfrak{o}, \alpha}$ for $(\mathfrak{o}, \alpha) \neq (\{0\}, 0)$. If $P = 0$ and $(\mathfrak{o}, \alpha) \neq (\{0\}, 0)$, we have the following finiteness uniformly for ζ :

$$\int_{T \times U} |\kappa^i(\zeta)_{0, \mathfrak{o}, \alpha}|_{h_1}^2 \text{dvol} \leq C_1 \int_{T \times U} |\kappa^i(\zeta)_{0, \mathfrak{o}, \alpha}|_{h_2}^2 |w|^{-2-\epsilon} \text{dvol} < \infty$$

We have $|g_{0, \{0\}, 0}|_{h_1} \leq C_1|w|^{-1}$ for some C_1 . We take a sufficiently small $C_2 > 0$, and we put $H_\zeta := \{w \mid |w|^{-1} < C_2|\zeta|\}$. We can find a unique family of C^∞ -sections $\mu(\zeta)$ of $\mathcal{P}_{-1}E \otimes \Omega_{T \times \mathbb{P}^1}^{0,1} \otimes L_\zeta^{-1}$ on H_ζ such that

$$(\bar{\partial}_E + \zeta d\bar{z})\mu(\zeta) = \left(\kappa^1(\zeta)_{0, \{0\}, 0} d\bar{z} + \kappa^2(\zeta)_{0, \{0\}, 0} d\bar{w} \right)_{|H_\zeta}.$$

There exists $C_3 > 0$ such that the following holds:

$$\int_{T \times \{w\}} |\mu(\zeta)|_{h_2}^2 |dzd\bar{z}| \leq C_3|\zeta|^{-2} \int_{T \times \{w\}} |\kappa^1(\zeta)_{0, \{0\}, 0}|_{h_2}^2 |dzd\bar{z}|$$

Let $\chi(w)$ be a $\mathbb{R}_{\geq 0}$ -valued C^∞ -function such that $\chi(w) = 1$ if $|w|^{-1} \leq C_2/4$ and $\chi(w) = 0$ if $|w|^{-1} \geq C_2/2$. We set

$$\begin{aligned} \tilde{\kappa}^1(\zeta) &= \kappa^1(\zeta)_{0, \{0\}, 0} - \partial_{\bar{z}}(\chi(w\zeta)\mu(\zeta)) = (1 - \chi(w\zeta))\kappa^1(\zeta)_{0, \{0\}, 0} \\ \tilde{\kappa}^2(\zeta) &= \kappa^2(\zeta)_{0, \{0\}, 0} - \partial_{\bar{w}}(\chi(w\zeta)\mu(\zeta)) = (1 - \chi(w\zeta))\kappa^2(\zeta)_{0, \{0\}, 0} - (\partial_{\bar{w}}\chi)(w\zeta) \cdot \zeta \cdot \mu(\zeta) \end{aligned}$$

For any $\delta > 0$, we have the following finiteness, which is uniform for ζ :

$$\int_{T \times U} (|\tilde{\kappa}_1(\zeta)|_{h_2}^2 + |\tilde{\kappa}_2(\zeta)|_{h_2}^2) |dzd\bar{z}| \frac{|dw d\bar{w}|}{|w|^{2+\delta}} \leq C_{1, \delta}$$

Hence, we have the following for any $\delta > 0$:

$$\int_{T \times U} (|\tilde{\kappa}_1(\zeta)|_{h_1}^2 + |\tilde{\kappa}_2(\zeta)|_{h_1}^2) |dzd\bar{z}dw d\bar{w}| \leq C_{2, \delta} \int_{T \times U} (|\tilde{\kappa}_1(\zeta)|_{h_2}^2 + |\tilde{\kappa}_2(\zeta)|_{h_2}^2) |dzd\bar{z}| \frac{|dw d\bar{w}|}{|w|^{2+\delta}} |\zeta|^{-2\delta} \leq C_{3, \delta} |\zeta|^{-2\delta}$$

Hence, we obtain $|t(\zeta)|_h \leq C_{4\epsilon} |\zeta|^{-\delta}$ for any $\delta > 0$. Thus, the proof of Lemma 6.17 is finished. ■

Let us prove $\text{Nahm}_*(\mathcal{P}_*E_1) = \mathcal{P}_*E$. We have the following, which is similar to Lemma 6.14.

Lemma 6.18 *We have only to show $\text{Nahm}_a(\mathcal{P}_*E_1) \subset \mathcal{P}_aE$ for any a .* ■

Around each $P \in D$, we have the decomposition

$$\text{Nahm}_*(\mathcal{P}_*E_1) = \bigoplus_{\mathfrak{o}, \alpha} \text{Nahm}_*(\mathcal{P}_*E_1)_{P, \mathfrak{o}, \alpha}, \quad (90)$$

according to the decomposition (88). We have only to prove $\text{Nahm}_a(\mathcal{P}_*E_1)_{P, \mathfrak{o}, \alpha} \subset \mathcal{P}_a(E)$. We shall argue the case $P = 0$ in the following. The other case can be proved similarly. We shall omit the subscript P . We take a small neighbourhood U_ζ of P .

Let us consider the case $(\mathfrak{o}, \alpha) \neq (0, 0)$. Let $U_\tau \subset \mathbb{P}^1$ be a small neighbourhood of ∞ with the coordinate $\tau = w^{-1}$. Take $\mathfrak{a} \in \mathfrak{o}$. For each $c \in \mathbb{R}$, we have the frame of Lemma 5.25. We have only to show

$$|[\tau_\bullet^j v_{\mathfrak{o}, i}]|_h = O\left(|\zeta|^{(b-j+p_\bullet-m_\bullet/2)(p_\bullet-m_\bullet)^{-1}}\right). \quad (91)$$

Here, b is the parabolic degree of $v_{\mathfrak{o}, i}$.

We give a preliminary. We take a ramified covering $U_u \rightarrow U_\zeta$ given by $\zeta = u^{p_\bullet-m_\bullet}$. We put $G(\tau_\bullet) := \partial_w \mathfrak{a}(\tau_\bullet) - \alpha p_\bullet \tau_\bullet^{p_\bullet} = \sum_{j=0}^{m_\bullet} \beta_j \tau_\bullet^{p_\bullet-j}$. We take a complex number γ such that $\beta_{m_\bullet} \gamma^{p_\bullet-m_\bullet} - 1 = 0$. If U_ζ is sufficiently small, we can take holomorphic functions $\eta_0^{(i)}(u)$ ($i = 1, \dots, p_\bullet - m_\bullet$) on U_ζ satisfying

$$G(\eta_0^{(i)}(u)) - u^{p_\bullet-m_\bullet} = 0, \quad \lim_{u \rightarrow 0} u^{-1} \eta_0^{(i)}(u) = \gamma \exp(2\pi\sqrt{-1}i/(p_\bullet - m_\bullet))$$

There exist $C_1 > 0$ and $\epsilon_1 > 0$ such that $|u^{-1} \eta_0^{(i)}(u) - \gamma \exp(2\pi\sqrt{-1}i/(p_\bullet - m_\bullet))| \leq C_1 |u|^{\epsilon_1}$. The following lemma is similar to Lemma 6.15.

Lemma 6.19 *Let Z_u denote the support of $\text{Cok}(g_{\mathfrak{a}, \alpha} - u^{p_\bullet-m_\bullet})$ on $U_{\tau_\bullet, u}$. If U_τ and U_ζ are sufficiently small, we have a decomposition $Z_u = \coprod_{i=1}^{p_\bullet-m_\bullet} Z_u^{(i)}$ and positive constants C and ϵ such that $|\eta_0^{(i)}(u) - \eta_1| \leq C|u|^{1+m_\bullet+\epsilon}$ for any $\eta_1 \in Z_u^{(i)}$.* ■

We set $d := 1 + m_\bullet/2$. We consider the following sections of $E_{\mathfrak{a}}^\bullet \otimes \Omega^{0,1}$:

$$\begin{aligned} \mu_1(v_{\mathfrak{o}, i}, u) &:= \rho(|u|^d(\tau_\bullet - \eta_0(u))) v_{\mathfrak{o}, i} d\bar{z} \\ \mu_2(v_{\mathfrak{o}, i}, u) &:= (g_{\mathfrak{a}, \alpha} - u^{p_\bullet-m_\bullet})^{-1} \left(\bar{\partial} \rho(|u|^d(\tau_\bullet - \eta_0(u))) \right) v_{\mathfrak{o}, i} \end{aligned}$$

The tuple $\boldsymbol{\mu}(v_{\mathfrak{o}, i}, u) := (\mu_1(v_{\mathfrak{o}, i}, u), \mu_2(v_{\mathfrak{o}, i}, u))$ induces a holomorphic section of $\text{Nahm}(\mathcal{P}_*E_1)_{P, \mathfrak{o}, \alpha}$. By Lemma 6.19, $\rho(|u|^d(\tau_\bullet - \eta_0(u)))$ is constantly 1 around Z_u . Hence, $\boldsymbol{\mu}(v_{\mathfrak{o}, i}, u)$ induces $[v_{\mathfrak{o}, i}]$.

By an elementary change of variables, we obtain the following for any $\delta > 0$:

$$\begin{aligned} \int |\mu_1(v_{\mathfrak{o}, i}, u)|_h^2 &\leq \int |\rho(|u|^d(\tau_\bullet - \eta_0(u)))|^2 |\tau_\bullet|^{-2(b+\delta)} |dz d\bar{z}| |dw d\bar{w}| \\ &\leq C_{1\delta} |u|^{-2(b+\delta)-2p_\bullet-2+2d} = C_{1\delta} |u|^{-2(b+\delta+p_\bullet-m_\bullet/2)} \end{aligned} \quad (92)$$

We also have the following:

$$\begin{aligned} \int |\mu_2(v_{\mathfrak{o}, i}, u)|_h^2 &\leq \int \int |\bar{\partial} \rho(|u|^d(\tau_\bullet - \eta_0(u)))|^2 |\tau_\bullet|^{-2(b+\delta)} \frac{1}{|\partial_w \mathfrak{a}(\zeta_\bullet) - \alpha \zeta_\bullet^{p_\bullet} - u^{p_\bullet-m_\bullet}|^2} \\ &\leq C_{2\delta} |u|^{-2(b+\delta)-2(p_\bullet-m_\bullet-1)-2d} = C_{2\delta} |u|^{-2(b+\delta+p_\bullet-m_\bullet/2)} \end{aligned} \quad (93)$$

Hence, we obtain the estimate (91).

Let us consider the case $(\mathfrak{o}, \alpha) = (\{0\}, 0)$. Note that $N(\mathcal{P}_{-1}E_1)_{0, \{0\}, 0} = N(\mathcal{U})_{0, \{0\}, 0} \subset \text{Nahm}_0(\mathcal{P}_*E_1)$. Let $\nu \in \text{Nahm}_{1+c}(\mathcal{P}_*E_1)_{0, \{0\}, 0} / N(\mathcal{P}_{-1}E_1)_{0, \{0\}, 0}$ for $-1 < c \leq 0$. We take $v \in \mathcal{P}_c V_{0, \{0\}, 0}$ which represents ν . We

naturally regard v as a C^∞ -section of $\mathcal{P}_c(E_1)_0$. Fix a sufficiently small number $b > 0$, and let ρ be a $\mathbb{R}_{\geq 0}$ -valued C^∞ -function on \mathbb{C}_τ such that $\rho(\tau) = 1$ if $|\tau| \leq b/2$ and $\rho(\tau) = 0$ if $|\tau| \geq b$. We obtain a C^∞ -section $\bar{\partial}(\rho(\tau)v d\bar{z})$ of $\mathcal{P}_{-1}(E_1)_0 \otimes \Omega^{0,2}$. By using $H^2(T \times \mathbb{P}^1, \mathcal{U} \otimes L_{-\zeta}) = 0$ for any ζ , we can take a holomorphic family of C^∞ -forms $\kappa(\zeta) = \kappa^1(\zeta)d\bar{z} + \kappa^2(\zeta)d\bar{w}$ of $\mathcal{U} \otimes \Omega^{0,1}$ such that $\bar{\partial}_{E \otimes L_{-\zeta}} \kappa(\zeta) = \bar{\partial}(\rho(\tau)v d\bar{z})$. Then, $\rho(\tau)v d\bar{z} - \kappa(\zeta)$ induces a holomorphic section \tilde{v} of $\text{Nahm}_{1+c}(\mathcal{P}_*E_1)$ around P which induces ν in $\text{Nahm}_{1+c}(\mathcal{P}_*E_1)/N(\mathcal{U})$.

We consider the following sections:

$$\mu_1(v, \zeta) := (\rho(\tau) - \rho(\zeta^{-1}\tau))v d\bar{z}$$

$$\mu_2(v, \zeta) := \bar{\partial}(\rho(\tau) - \rho(\zeta^{-1}\tau))(g_{0, \{0\}, 0} - \zeta)^{-1}(v)$$

Then, $\mu_1(v, \zeta) + \mu_2(v, \zeta) - \kappa(\zeta)$ induces the same section \tilde{v} .

We have the following for any $\delta > 0$:

$$\int |\mu_1|_{h_1}^2 |dw d\bar{w}| \leq C_\delta \int_{|\tau| \geq A|\zeta|} |\tau|^{-2(c+\delta)-4} |d\tau d\bar{\tau}| \leq C_\delta |\zeta|^{-2(c+1+\delta)}$$

We also have the following:

$$\int |\mu_2|_{h_1}^2 |dz d\bar{z}| \leq C_\delta \int |\bar{\partial}\rho(\zeta^{-1}\tau)|^2 |\zeta|^{-2} |\tau|^{-2(c+\delta)} \leq C_\delta |\zeta|^{-2(c+1+\delta)}$$

Because the support of $\bar{\partial}(\rho(\tau)v d\bar{z})$ is compact, we obtain $\int |\kappa|_{h_1}^2 \text{dvol} = O(|\zeta|^{-\delta})$ for any $\delta > 0$, by the argument in the proof of Lemma 6.17. We obtain $|\tilde{v}|_h \leq C_\delta |\zeta|^{-(c+1+\delta)}$ for any $\delta > 0$. Thus, we obtain $\text{Nahm}_{1+c}(\mathcal{P}_*E_1)_{0, \{0\}, 0} \subset \mathcal{P}_{1+c}E$, and the proof of Theorem 6.12 is finished. \blacksquare

6.3 Kobayashi-Hitchin correspondence for L^2 -instantons

6.3.1 Statements

Let \mathcal{P}_*E_1 be a good filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$ of degree 0 satisfying the conditions **(A3)**. (See §5.4.5 for good filtered bundles.)

Proposition 6.20 *\mathcal{P}_*E_1 is stable, if and only if $\text{Nahm}_*(\mathcal{P}_*E_1)$ is a stable filtered Higgs bundle. (See §5.3.1 for the stability condition of \mathcal{P}_*E_1 .)*

Before going to the proof, we give a consequence.

Theorem 6.21 *We set $E_1 := (\mathcal{P}_a E_1)_{|T \times \mathbb{C}}$. There exists a Hermitian-Einstein metric h of E_1 on $T \times \mathbb{C}$ such that (i) its curvature is L^2 with respect to h and the Euclidean metric, (ii) it is adapted to the filtered bundle \mathcal{P}_*E_1 . Such a metric is unique up to the multiplication of a positive constant.*

Proof By Proposition 6.20, $\text{Nahm}(\mathcal{P}_*E_1)$ is stable. By Corollary 5.22, we have $\deg \text{Nahm}(\mathcal{P}_*E_1) = \deg(\mathcal{P}_*E_1) = 0$. By Corollary 5.30, $\text{Nahm}(\mathcal{P}_*E_1)$ is a good filtered Higgs bundle. Hence, by the Kobayashi-Hitchin correspondence for wild harmonic bundles on curves [7], we obtain an adapted harmonic metric for $\text{Nahm}(\mathcal{P}_*E_1)$. Its Nahm transform induces a Hermitian-Einstein metric of E_1 adapted to the filtered bundle \mathcal{P}_*E_1 , by Theorem 6.11 and Proposition 5.21. \blacksquare

Remark 6.22 *This proof of Theorem 6.21 is based on the idea mentioned in Remark 5.13 of [8].* \blacksquare

6.3.2 Proof of Proposition 6.20

Let us prove the “if” part in Proposition 6.20. Suppose $\text{Nahm}(\mathcal{P}_*E_1)$ is stable. By the Kobayashi-Hitchin correspondence for wild harmonic bundles on curves [7] (see also [38] for the case of good filtered flat bundles), we have an adapted harmonic metric for $\text{Nahm}(\mathcal{P}_*E_1)$. By Theorem 6.11, its Nahm transform gives an adapted Hermitian-Einstein metric for \mathcal{P}_*E_1 . By Corollary 4.4, \mathcal{P}_*E_1 is polystable. If it is not stable, the decomposition

into the stable components induces a decomposition of $\text{Nahm}(\mathcal{P}_*E_1)$, which contradicts with the stability of $\text{Nahm}(\mathcal{P}_*E_1)$. Hence, \mathcal{P}_*E_1 is stable.

Let us show the “only if” part in Proposition 6.20. Let $(\mathcal{P}_*E, \theta) := \text{Nahm}(\mathcal{P}_*E_1)$. Let $(\mathcal{P}_*E', \theta') \subset (\mathcal{P}_*E, \theta)$ be a strict filtered Higgs subbundle. We obtain a subcomplex $\tilde{\mathcal{C}}^\bullet(\mathcal{P}_*E', \theta') \subset \tilde{\mathcal{C}}^\bullet(\mathcal{P}_*E, \theta)$ on $T^\vee \times T \times \mathbb{P}^1$. Let $\tilde{\mathcal{Y}}^\bullet = (\tilde{\mathcal{Y}}^0 \rightarrow \tilde{\mathcal{Y}}^1)$ be the quotient complex.

Lemma 6.23 *The induced morphism $R^1p_{23*}(\tilde{\mathcal{C}}^\bullet(\mathcal{P}_*E', \theta')) \rightarrow R^1p_{23*}(\tilde{\mathcal{C}}^\bullet(\mathcal{P}_*E, \theta))$ is injective.*

Proof By the construction, $\tilde{\mathcal{Y}}^0$ is locally free. Hence, we obtain that $R^0p_{23*}\tilde{\mathcal{Y}}^0$ is torsion-free. Because

$$R^0p_{23*}(\tilde{\mathcal{Y}}^\bullet) \rightarrow R^0p_{23*}\tilde{\mathcal{Y}}^0$$

is injective, we obtain that $R^0p_{23*}\tilde{\mathcal{Y}}^\bullet$ is torsion-free. On a small neighbourhood $U \subset \mathbb{P}^1$ of ∞ , we have $R^ip_{23*}(\tilde{\mathcal{C}}^\bullet(\mathcal{P}_*E', \theta')) = R^ip_{23*}(\tilde{\mathcal{C}}^\bullet(\mathcal{P}_*E, \theta)) = 0$ unless $i = 1$. It is easy to check that

$$R^1p_{23*}(\tilde{\mathcal{C}}^\bullet(\mathcal{P}_*E', \theta'))|_{T \times \{\infty\}} \rightarrow R^1p_{23*}(\tilde{\mathcal{C}}^\bullet(\mathcal{P}_*E, \theta))|_{T \times \{\infty\}}$$

is injective. Hence, $R^1p_{23*}(\tilde{\mathcal{C}}^\bullet(\mathcal{P}_*E', \theta'))|_{T \times U} \rightarrow R^1p_{23*}(\tilde{\mathcal{C}}^\bullet(\mathcal{P}_*E, \theta))|_{T \times U}$ is injective. Because

$$0 \rightarrow R^0p_{23*}\tilde{\mathcal{Y}}^\bullet \rightarrow R^1p_{23*}(\tilde{\mathcal{C}}^\bullet(\mathcal{P}_*E', \theta')) \rightarrow R^1p_{23*}(\tilde{\mathcal{C}}^\bullet(\mathcal{P}_*E, \theta))$$

is exact, we obtain $R^0p_{23*}\tilde{\mathcal{Y}}^\bullet = 0$, and $R^1p_{23*}(\tilde{\mathcal{C}}^\bullet(\mathcal{P}_*E', \theta')) \rightarrow R^1p_{23*}(\tilde{\mathcal{C}}^\bullet(\mathcal{P}_*E, \theta))$ is injective. \blacksquare

We define the parabolic structure of $R^1p_{23*}(\tilde{\mathcal{C}}^\bullet(\mathcal{P}_*E', \theta'))$ as in §5.2.1. The filtered bundle is denoted by $\mathcal{P}_*\mathcal{V}_1$. We have a naturally defined injective morphism $\mathcal{P}_*\mathcal{V}_1 \rightarrow \mathcal{P}_*E_1$. Hence, we have $\deg(\mathcal{P}_*\mathcal{V}_1) \leq 0$. By the argument in 5.2.2, we obtain

$$\int_{T \times \mathbb{P}^1} c_1(\mathcal{P}_*\mathcal{V}_1)\omega_T - \int_{T \times \mathbb{P}^1} c_1(R^2p_{23*}\tilde{\mathcal{C}}^\bullet(\mathcal{P}_*E', \theta'))\omega_T = \deg(\mathcal{P}_*E')$$

Because $R^2p_{23*}\tilde{\mathcal{C}}^\bullet(\mathcal{P}_*E', \theta')$ is a torsion sheaf, we obtain $\int_{T \times \mathbb{P}^1} c_1(R^2p_{23*}\tilde{\mathcal{C}}^\bullet(\mathcal{P}_*E', \theta'))\omega_T \geq 0$. Hence, we obtain $\deg(\mathcal{P}_*E') \leq 0$, i.e., (\mathcal{P}_*E, θ) is semistable.

We have $(\mathcal{P}_*E', \theta') \subset (\mathcal{P}_*E, \theta)$ such that $(\mathcal{P}_*E', \theta')$ is stable of degree 0. If $(\mathcal{P}_*E', \theta')$ has no singularity, it is isomorphic to \mathcal{O}_T^\vee with a Higgs field $\alpha d\zeta$ ($\alpha \in \mathbb{C}$), and hence $R^1p_{23*}\tilde{\mathcal{C}}^\bullet(\mathcal{P}_*E', \theta')$ is a non-zero torsion subsheaf of E_1 , which contradicts with Lemma 6.23. Therefore, $(\mathcal{P}_*E', \theta')$ has a singularity, and $\text{Nahm}_*(\mathcal{P}_*E', \theta') \neq 0$ is a good filtered subbundle of \mathcal{P}_*E_1 . By the stability of \mathcal{P}_*E_1 , we have $\text{rank Nahm}_*(\mathcal{P}_*E', \theta') = \text{rank } E_1$. Because $\deg \text{Nahm}_*(\mathcal{P}_*E', \theta') = \deg(\mathcal{P}_*E)$, we have $\text{Nahm}_*(\mathcal{P}_*E', \theta') = \mathcal{P}_*E_1$ in codimension one. Because both of them are filtered bundles, we have $\text{Nahm}_*(\mathcal{P}_*E', \theta') = \mathcal{P}_*E_1$ on $T \times \mathbb{P}^1$. Then, we obtain $(\mathcal{P}_*E', \theta') = (\mathcal{P}_*E, \theta)$ by the involutivity of the algebraic Nahm transforms. \blacksquare

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Address

*Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan,
takuro@kurims.kyoto-u.ac.jp*